1. Homeomorphism Classification of Simply Connected Compact 4-Manifolds

**Theorem 1** (Freedman). $M_1, M_2$ compact, simply connected, oriented smooth 4-manifolds are homeomorphic $\iff$ the intersection pairings $Q_i : H_2(M_i, \mathbb{Z}) \times H_2(M_i, \mathbb{Z}) \to \mathbb{Z}$ are isomorphic as integer quadratic forms (of $|\det| = 1$). It suffices to check that the following invariants coincide: $b_2 = \text{rk} H_2(M), \sigma = b_2^+ - b_2^-$ (the signature), and $Q(x, x) \mod 2\pi x$ (the parity).

**Example.** For $M = \mathbb{C}P^2$, we have $Q_{\mathbb{C}P^2} = (1)$ on $H_2(\mathbb{C}P^2, \mathbb{Z}) = \mathbb{Z}[\mathbb{C}P^2]$, while $N = \overline{\mathbb{C}P^2}$ (with reversed orientation) has $Q_{\overline{\mathbb{C}P^2}} = (-1)$. By Mayer-Vietoris, one can see that $H_2(M\#N) = H_2(M) \oplus H_2(N)$ and $Q_{M\#N} = Q_M \oplus Q_N$. Applying this to $m$ copies of $\mathbb{C}P^2$ and $n$ copies of $\overline{\mathbb{C}P^2}$ gives the matrix

$$
\begin{pmatrix}
I_{m \times m} & -I_{n \times n}
\end{pmatrix}
$$

which gives all the unimodular odd quadratic forms ($|\det| = 1$).

**Example.** $Q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has $b_2^+ = b_2^- = 1$ like $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, but different parity.

**Example.** A K3 is a surface of degree 4 in $\mathbb{C}P^3$ (given, for instance, by $\{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}$). We have $b_2 = 22, b_2^+ = 3, b_2^- = 19$, and $Q = 2.(-E_8) \oplus 3.\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where $(-E_8)$ is the matrix

(1)

$$
\begin{pmatrix}
-2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & 0 & -2
\end{pmatrix}
$$

**Theorem 2** (Donaldson). In the even case, $Q = (2k).(+E_8) \oplus m.\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

**Remark.** The Rokhlin signature theorem ($16|\sigma$ in the even case) implies that the number of $\pm E_8$ summands is even.

**Remark.** The $\frac{11}{4}$ conjecture claims that the $m$ in the theorem above satisfies $m \geq 3k$: it has been shown (Furuta, 1995) that $m \geq 2k$.

**Remark.** Smooth compact 4-manifolds may have infinitely many exotic smooth structures: K3 surfaces are known to have infinitely many smooth structures, as do the manifolds $\mathbb{C}P^2 \# n.\overline{\mathbb{C}P^2}$ for $n \geq 3$.

2. Seiberg-Witten Invariants [J. Morgan], [Witten ‘94]

Let $X^4$ be a compact manifold, with Riemannian metric $g$ and spin$^c$ structure $s$. The goal of Seiberg-Witten theory is to assign a number to the pair $(g, s)$ giving the number of “abelian supersymmetric magnetic monopoles” on the manifold.

**Definition 1.** A spin$^c$ structure is a rank 4 Hermitian complex vector bundle $S \to X$ along with a Clifford multiplication (unitary representation of a Clifford algebra) $\gamma : T^* X \times S \to S$ (i.e. $\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2\langle u, v \rangle \text{id}$ and $\gamma(u)^* = -\gamma(u)$).
Example. For \{e_i\} an orthonormal basis, \(\gamma(e^i) \in U(S)\), since \(\gamma(e^i)^2 = -1\), and \(\gamma(e^i)\gamma(e^j) + \gamma(e^j)\gamma(e^i) = 0\).

We extend our Clifford multiplication to
\[
\gamma : \bigwedge (T^*X) \times S \to S, \gamma(e^{i_1} \wedge \cdots \wedge e^{i_r}) = \gamma(e^{i_1}) \cdots \gamma(e^{i_r})
\]
for (\(e^i\)) orthonormal. Applying this to the volume form gives \(\gamma(\text{vol})^2 = (\gamma(e^1)\gamma(e^2)\gamma(e^3)\gamma(e^4))^2 = \text{id}\) and thus a decomposition \(S = S^+ \oplus S^-\), with the former having \(\gamma(\text{vol}) = -1\) and the latter \(\gamma(\text{vol}) = 1\). Moreover, \(\gamma(e^i)\) maps \(S^\pm\) to \(S^\mp\).

Lemma 1. We can represent complexified forms via \(\gamma : \bigwedge \otimes \mathbb{C} \xrightarrow{\sim} \text{End}(S^+ \oplus S^-)\). More specifically, we have decompositions
\[
\bigwedge^{\text{even}} \otimes \mathbb{C} \cong \text{End}(S^+) \oplus \text{End}(S^-)
\]
\[
\bigwedge^{\text{odd}} \otimes \mathbb{C} \cong \text{Hom}(S^+, S^-) \oplus \text{Hom}(S^-, S^+)
\]
with \(\gamma(*\alpha) = \gamma(\alpha)\) on \(S^+\) and \(-\gamma(\alpha)\) on \(S^-\) for any \(\alpha \in \bigwedge^2\), so
\[
\text{End}(S^+) = \mathbb{C} \oplus (\bigwedge^+ \otimes \mathbb{C}), \text{End}(S^-) = \mathbb{C} \oplus (\bigwedge^- \otimes \mathbb{C})
\]

Theorem 3. Every compact 4-manifold admits spin\(^c\) structures classified up to 2-torsion by \(c = c_1(S^+) = c_1(S^-) = c_1(L) \in H^2(X, \mathbb{Z})\), where \(L = \text{det}(S^+) = \bigwedge^2 S^+ = \bigwedge^2 S^+\). Moreover, \(c\) is a characteristic element, i.e. \((c_1(L), \alpha) \equiv Q(\alpha, \alpha) \mod 2\).

In particular, if \(E \to X\) is a line bundle, the mapping \((S^+, S^-) \to (S^+ \otimes E, S^- \otimes E)\) gives a twisting of the spin\(^c\) structure by any line bundle.

Proposition 1. If \(X\) admits a \(g\)-orthogonal almost-complex structure \(J\), then \(\exists\) a canonical spin\(^c\) structure given by \(S^+ = \bigwedge^{0,0} \oplus \bigwedge^{0,2}, S^- = \bigwedge^{0,1}\) with
\[
\forall u \in T^*X, \gamma(u) = \sqrt{2}[u^{0,1} \cdot - \iota(u^{0,1}) \cdot (\cdot)]
\]
Note that \(L = \bigwedge^2 S^- = \bigwedge^2 S^+ = \bigwedge^0 S^+\) is the anti-canonical bundle. All other spin structures are given by \(S^+ = E \oplus (\bigwedge^{0,2} \otimes E), S^- = \bigwedge^{0,1} \otimes E, \forall E \to X\) a line bundle.

3. Dirac Operator

Definition 2. A spin\(^c\) connection on \(S^\pm\) is a Hermitian connection \(\nabla^A\) s.t.
\[
\nabla^A_\upsilon(\gamma(u)\phi) = \gamma(\nabla^L_\upsilon u)\phi + \gamma(u)\nabla^A_\upsilon \phi
\]

Proposition 2. Any two spin\(^c\) connections differ by a 1-form on \(X\) of the type \(i\alpha \otimes \text{id}_{S^\pm}\), and the induced connection \(A\) on \(L = \bigwedge^2 S^\pm\) defines the spin\(^c\) connection uniquely.

Definition 3. Given a spin\(^c\) structure and a connection, the Dirac operator is
\[
D_A : \Gamma(S^\pm) \to \Gamma(S^\pm), D_A\psi = \sum_i \gamma(e^i)\nabla^A_{e^i} \psi
\]
for \(\{e_i\}\) an orthonormal basis (though it is independent of choice of basis).

Example. On a Kähler manifold, \(S^+ = E \oplus \bigwedge^{0,2} \otimes E, S^- = \bigwedge^{0,1} \otimes E\), \(\nabla^A\) corresponds to a unitary connection \(\nabla^u\) on \(E\), i.e. via \(\nabla^A = \nabla^L \otimes 1 + 1 \otimes \nabla^u\). Then \(D_A = \sqrt{2} |\partial a + \bar{\partial} a|\) and \(D_A^2 = 2|\partial a|^2\).

Definition 4. The Seiberg-Witten equations are the equations
\[
D_A\psi = 0 \in \Gamma(S^-), \quad \gamma(F_A^+ \upsilon) = (\psi^* \otimes \psi)_0 \in \Gamma(\text{End}(S^+))
\]
where \(A\) is a Hermitian connection on \(L = \bigwedge^2 S^\pm\) (corresponding to a spin\(^c\) connection on \(S^\pm\)), \(\psi \in \Gamma(S^-)\) is a section, \(F_A^+ = \frac{1}{2}(F_A + *F_A) \in \Omega^2_+\) for \(F_A \in \Omega^2\) the curvature of \(A\), and \((\psi^* \otimes \psi)_0 = \psi^* \otimes \psi - \frac{1}{2} |\psi|^2\) is the traceless part of \(\psi^* \otimes \psi\).