1. Spin Structures

Let \((X^4, g)\) be an oriented Riemannian manifold, \(S = S_+ \oplus S_- \to X\) a spin\(^c\) structure with Clifford multiplication \(\gamma : T^* X \otimes S \to S\).

**Example.** If \(X\) is almost-complex, \(S_+ = (\bigwedge^0 \mathbb{C} \otimes \mathbb{C}) \oplus (\bigwedge^0 \mathbb{C} \otimes \mathbb{C})\), \(S_- = (\bigwedge^1 \mathbb{C} \otimes \mathbb{C})\), \(\gamma(u) = \sqrt{2}[u^{0,1} \wedge \cdot - i_{(u^1, 0)}] \) . As defined last time, \(L = \det(S_+) = \det(S_-) = K_X^{-1} \otimes \mathbb{C}^2\).

As we stated last time, the Clifford multiplication extends to differential forms with \(\bigwedge^2 T^* X \cong \operatorname{End}_{\text{TLS}}(S^+)\) (where the latter group is the space of traceless, anti-hermitian endomorphisms). We also have the Dirac operator associated to a spin\(^c\) connection \(\nabla^A\) on \(S\):

\[
D_A : \Gamma(S^+) \to \Gamma(S^+) , \quad D_A \psi = \sum_i \gamma(e^i)(\nabla_{e^i}^A \psi)
\]

**Example.** If \(X\) is Kähler, the spin\(^c\) connection is induced by \(\nabla_a\) connection on \(E\), and \(D_A = \sqrt{2}(\overline{\partial} + \partial^*)\).

**Example.** \(\nabla^A = \nabla^A_a + ia \otimes \text{id}\) on \(S_\pm\) for \(a \in \Omega^1\) corresponding to \(A = A_0 + 2ia\) on \(L\). The associated decomposition of the Dirac operator is \(D_A = D_{A_0} + \gamma(a)\).

2. Seiberg-Witten Equations

**Definition 1.** The Seiberg-Witten equations are the equations

\[
D_A \psi = 0 \in \Gamma(S^-) \\
\gamma(F_A^+) = (\psi^* \otimes \psi)_0 \in \Gamma(\operatorname{End}(S^+))
\]

where \(A\) is a Hermitian connection on \(L = \bigwedge^2 S^\pm\) (corresponding to a spin\(^c\) connection \(\nabla^A\)), \(\psi \in \Gamma(S^+)\) is a section, \(F_A^+ = \frac{1}{2}(F_A + *F_A) \in i\Omega^2\) for \(F_A \in \Omega^2\) the curvature of \(A\), \((\psi^* \otimes \psi)_0\) is the traceless part of \(\psi^* \otimes \psi\), and \(\mu\) is an imaginary self-dual form fixed in advance.

Now, there exists an \(\infty\)-dimensional group of symmetries preserving solutions, called the gauge group \(\mathcal{G} = C^\infty(X, S^1)\) where \(f \in C^\infty(X, S^1)\) acts by

\[
(A, \psi) \mapsto (A - 2df \cdot f^{-1}, f \psi)
\]

**Proposition 1.** This preserves the solution space, and the action of \(\mathcal{G}\) is free unless \(\psi \equiv 0\) (reducible solutions), where \(\text{Stab}((A, 0)) \cong S^1\) is the space of constant maps.

Reducible solutions can happen \(\iff F_A^+ = \mu\) has a solution \(\iff (g, \mu)\) lie in a codimension \(b_2^+\) subspace. Thus, we want to assume \(b_2^+(X) \geq 1\), and \((g, \mu)\) generic. Note that, for \(\mu = 0\), \(F_A^+ = 0 \Rightarrow \frac{i}{2\pi} F_A\) is closed and antiselfdual in the class \(c_1(L) \in \mathcal{H}_2 \subseteq \mathcal{H}_2^- \oplus \mathcal{H}_2^+ = \mathcal{H}^2\).

**Definition 2.** The moduli space of solutions \(\mathcal{M}(S, g, \mu)\) is the set of solutions modulo \(\mathcal{G}\).

**Theorem 1.** For \((g, \mu)\) generic, \(\mathcal{M}\) (if nonempty) is a smooth, compact, orientable manifold of dimension

\[
d(S) = \frac{1}{4}(c_1(L)^2 \cdot [X] - (2\chi + 3\sigma))
\]
Idea: We want to understand, given a solution \((A_0, \psi_0)\) to the SW equations, the nearby solutions to the same equations. We linearize the SW equations, and let \((a, \phi) \in \Omega^1(X, i\mathbb{R}) \times C^\infty(S^+)\) be a small change in the solution, obtaining
\[
(5) \quad P_1 : (a, \phi) \mapsto D_{A_0} \phi + \gamma(a) \psi_0
\]
as the linearization of the first equation and
\[
(6) \quad P_2 : (a, \phi) \mapsto \gamma((da)^+) - (\phi \otimes \psi_0^* + \psi_0 \otimes \phi^*)_0
\]
as the linearization of the second equation. We restrict \(P = P_1 \oplus P_2\) to a slice transversal to the \(G\)-action \(A \mapsto A - 2df \cdot f^{-1}, \psi \mapsto f \psi\), i.e. to \(S = \{(a, \phi)|da = 0 \text{ and } \text{Im}((\phi, \psi_0)_{L^2}) = 0\}\) (which is transverse to the \(G\)-orbit at \((A_0, \psi_0)\)). Then \(P|_{\text{Ker } d^* X, i\mathbb{R}}\) is a differential operator of order 1, and is Fredholm (f.d. kernel and cokernel) since
\[
(7) \quad (P \oplus d^*) : L^2(X, i\mathbb{R}) \times L^2(S^+) \rightarrow L^2(S^-) \times L^2(X, i\mathbb{R})
\]
\((= D_{A_0} \oplus (d^* \oplus d^*) + \text{order } 0)\) is elliptic. Elliptic regularity implies that both \(\text{Ker}, \text{Coker}\) lie in \(C^\infty\). For generic \((g, \mu)\), \(P\) is surjective (specifically, consider \(\{(A, \psi, \mu)\}/G\) and apply Sard’s theorem to project to \(\mu\) and find a good choice). We expect that \(\text{Ker } P\) is the tangent space to \(\mathcal{M}\): this is only ok if \(\text{Coker } P = 0\), so we can use the implicit function theorem to show that \(\mathcal{M}\) is smooth with \(TM = \text{Ker } P|_S\). The statement about the dimension follows from the Atiyah-Singer index theorem, which gives a formula for \(d(S) = \text{ind}(P|_S) = \text{dim} \text{Ker } P - \text{dim} \text{Coker } P\). Compactness of \(\mathcal{M}\) follows from the a priori bounds on the solutions: the key point is that we get a bound on sup \(|\psi|\), so elliptic regularity and ”bootstrapping” give us bounds in all norms.

Consider a solution \((A, \psi)\) of the SW equations (for simplicity assume \(\mu = 0\)). We have the following Weitzenbock formula for the Dirac operator:
\[
(8) \quad D_A^2 \psi = \nabla_A^* \nabla_A \psi + \frac{s}{4} \psi + \frac{1}{2} \gamma(F_A^+) \psi
\]
where \(\nabla_A^*\) is the \(L^2\)-adjoint of \(\nabla_A\), \(s\) is the scalar curvature of the metric \(g\) (this can be shown by calculation in a local frame). Now,
\[
(9) \quad D_A \psi = 0 \implies 0 = \langle D_A^2 \psi, \psi \rangle = \langle \nabla_A^* \nabla_A \psi, \psi \rangle + \frac{s}{4} |\psi|^2 + \frac{1}{2} \gamma(F_A^+) \psi, \psi \rangle
\]
where \(\gamma(F_A^+) = (\psi^* \otimes \psi)_0 = \psi^* \otimes \psi - \frac{1}{2} |\psi|^2\). Then
\[
(10) \quad 0 = \frac{1}{2} d^* d |\psi|^2 + |\nabla_A \psi|^2 + \frac{s}{4} |\psi|^2 + \frac{1}{4} |\psi|^4
\]
Take a point where \(|\psi|\) is maximal. Then
\[
(11) \quad \frac{1}{2} d^* d |\psi|^2 \geq 0 \implies \frac{s}{4} |\psi|^2 + \frac{1}{4} |\psi|^4 \leq 0 \implies |\psi|^2 \leq \max(-s, 0)
\]

**Theorem 2.** If \(g\) has scalar curvature \(> 0\), then the SW-invariants \(\equiv 0\).

**Proof.** A small generic perturbation ensures that there are no reducible solutions. The above estimate on \(\sup |\psi|\) ensures that there are no irreducible solutions either. \(\square\)