11 Lecture 11 (Notes: K. Venkatram)

11.1 Integrability and spinors

Given $L \subset T \oplus T^*$ maximal isotropic, we get a filtration $0 \subset K_L = F^0 \subset F^1 \subset \cdots \subset F^n = \Omega^* (M)$ via $F^k = \{ \psi : \Lambda^{k+1} L \cdot \psi = 0 \}$. Furthermore, for $\phi \in K_L$, we have

$$X_1 X_2 d\phi = [[d, X_1], X_2] \phi = [X_1, X_2] \phi$$

(21)

for all $X_1, X_2 \in L$ (where $d = d_H$). Thus, in general, $d \phi \in F^3$, and $L$ is involutive $\Leftrightarrow$ $d \phi \in F^1$.

Now, assume $d(F^i) \subset F^{i+3}$ (and in $F^{i+1}$ if $L$ is integrable) $\forall i < k$ and $\psi \in F^k$. Then

$$[X_1, X_2] \psi = [[d, X_1], X_2] \psi = dX_1 X_2 \psi + X_1 dX_2 \psi - X_2 dX_1 \psi - X_2 X_1 d\psi$$

$$X_1 X_2 d\psi = -dX_1 X_2 \psi - X_1 dX_2 \psi + X_2 dX_1 \psi + [X_1, X_2] \psi$$

(22)

Note that, in the latter expression, each of the parts on the RHS have degree $(k - 1) + 2 = k + 1$, so $d \psi \in F^{k+1}$ if $L$ is integrable and $F^{k+3}$ otherwise.

Next, suppose that the Courant algebroid $E$ has a decomposition $L \oplus L'$ into transverse Dirac structures.

1. Linear algebra:

- $L' \cong L^*$ via $\langle \cdot, \cdot \rangle$.
- The filtration $K_L = F^0 \subset F^1 \subset \cdots \subset F^n$ of spinors becomes a $\mathbb{Z}$-grading $K_L \oplus (L' \cdot K_L) \oplus \cdots \oplus (\Lambda^k L' \cdot K_L) \oplus \cdots \oplus (\det L' \cdot K_L)$, i.e. $\bigoplus (\Lambda^k L^*) K_L$.

**Remark.** Note that $L' \cdot (\det L' \cdot K_L) = 0$, so $\det L' \cdot K_L = \det L^* \otimes K_L = K_{L'}$.

Thus, we have a $\mathbb{Z}$-grading $S = \bigoplus_{k=0}^n \mathcal{U}_k$.

- If the Mukai pairing is nondegenerate on pure spinors, then $K_L \otimes K_{L'} = \det T^*$.
2. Differential structure: via the above grading, we have $F^k(L) = \bigoplus_{i=0}^k \mathcal{U}_i$, $F^k(L') = \bigoplus_{i=0}^k \mathcal{U}_{n-i}$, so $d(\mathcal{U}_k) = d(F^k(L) \cap F_{n-k}(L'))$. By parity, $d\mathcal{U}_k \cap \mathcal{U}_k = 0$, so a priori
\[
d = (\pi_{k-3} + \pi_{k-1} + \pi_{k+1} + \pi_{k+3}) \circ d = T' + \partial' + \partial + T
\] (23)

**Problem.** Show that $T' : \mathcal{U}_k \rightarrow \mathcal{U}_{k-3}, T : \mathcal{U}_k \rightarrow \mathcal{U}_{k+3}$ are given by the Clifford action of tensors $T' \in \Lambda^3 L, T \in \Lambda^3 L^*.$

**Remark.** This splitting of $d = d_H$ can be used to understand the splitting of the Courant structure on $L \oplus L^*$. Specifically, $d^2 = 0 \implies$
\[
\begin{align*}
-4 & \quad T'\partial' + \partial'T' = 0 \\
-2 & \quad (\partial')^2 + T\partial + \partial T' \\
0 & \quad \partial\partial' + \partial'\partial + TT' + T'T \\
2 & \quad \partial^2 + T\partial' + \partial'T \\
4 & \quad T\partial + \partial T = 0
\end{align*}
\] (24)

11.2 Lie Bialgebroids and deformations

We can express the whole Courant structure in terms of $(L, L^*)$. Assume for simplicity that $L, L^*$ are both integrable, so $T = T' = 0$. Then

1. Anchor $\pi \rightarrow$ a pair of anchors $\pi : L \rightarrow T, \pi' : L' \rightarrow T$.

2. An inner product $\rightarrow$ a pairing $L' = L^*, \langle X + \xi, X + \xi \rangle = \xi(X)$.

3. A bracket $\rightarrow$ a bracket $[,]$ on $L, [,]_*$ on $L^*$. Specifically, for $x, y \in L, \phi \in \mathcal{U}_0$,
\[
[x, y]_\phi = [(d, x), y]_\phi = xyd\phi = x\phi(y + T)\phi = xyT\phi = (i_xi_yT)\phi
\] (25)

The induced action on $S$ is $d_L\alpha = [\partial, \alpha]$, giving us an action of $L$ on $L^*$ as $\pi_{L^*} [x, \xi]$ for $x \in L, \xi \in L^*$. Expanding, we have
\[
[x, \xi]_\phi = [(\partial, x), \xi]_\phi = \partial x\xi\phi + x\partial\xi\phi - \xi x\partial\phi - (i_x\xi)\partial\phi
= \partial (i_x\xi)\phi + x(d_L\xi)\phi - (i_x\xi)\partial\phi = (d_Li_x\xi + i_xd_L\xi)\phi = (L_x\xi)\phi
\] (26)

If $T = 0$, then $x \rightarrow L_x$ is an action (guaranteed by the Jacobi identity of the Courant algebroid). If $L, L'$ are integrable,
\[
L_x[\xi, \eta]_* = \pi_{L^*}[x, [\xi, \eta]] = \pi_{L^*}([\langle x, \xi, \eta \rangle] + [\xi, [x, \eta]])
\] (27)

**Problem.** This implies that $d[., .]_* = [d, .]_* + [., d\cdot]_*$.

As a result of these computations, we find that, for $X, Y \in L, \xi, \eta \in L^*$,
\[
[X + \xi, Y + \eta] = \langle X, Y \rangle + \langle X, \eta \rangle_L + \langle \xi, Y \rangle_L + \langle \xi, \eta \rangle_{L^*} + \langle [X, \xi], Y \rangle_{L^*} + \langle [\xi, Y], L^* \rangle_L
= \langle X, Y \rangle + L_\xi Y - i_\eta d_\cdot X + [\xi, \eta] + L_\xi \eta - i_\eta d\xi
\] (28)

There are no $H$ terms since we assumed $T = T' = 0$. Overall, we have obtained a correspondence between transverse Dirac structures $(L, L')$ and Lie bialgebroids $(L, L^*)$ with actions and brackets $L \rightarrow T, L^* \rightarrow T$ s.t. $d$ is a derivation of $[,]_*$.  

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Finally, we can deform the Dirac structure in pairs. Specifically, for \( \epsilon \in C^\infty(\bigwedge^2 L^*) \) a small \( B \)-transform, \( e^\epsilon(L) = L_\epsilon \), one can ask when \( L_\epsilon \) is integrable. We claim that this happens \( \Leftrightarrow d_L \epsilon + \frac{1}{2} [\epsilon, \epsilon]_\ast = 0 \). To see this, note that

\[
\langle [e^\epsilon x, e^\epsilon y], e^\epsilon z \rangle = \langle [e^\epsilon x, e^\epsilon y]_L, e^\epsilon z \rangle + \langle [e^\epsilon x, e^\epsilon y]_{L^*}, e^\epsilon z \rangle = (d_L \epsilon)(x, y, z) + \frac{1}{2} [\epsilon, \epsilon]_\ast(x, y, z)
\]

via an analogous computation to that of \( e^B T \) and \( e^{\pi T^*} \) from before.