2 Lecture 2 (Notes: A. Rita)

2.1 Comments on previous lecture

(0) The Poincaré lemma implies that the sequence

\[ \ldots \rightarrow C^\infty(\wedge^{k-1}T^*) \xrightarrow{d} C^\infty(\wedge^k T^*) \xrightarrow{d} C^\infty(\wedge^{k+1} T^*) \rightarrow \ldots \]

is an exact sequence of sheaves, even though it is not an exact sequence of vector spaces.

(1) We defined the Lie derivative of a vector field X to be \( L_X = [\iota_X, d] \). Since \( \iota_X \in \text{Der}^{-1}(\Omega^1(M)) \) and \( d \in \text{Der}^1(\Omega^1(M)) \), we have

\[ [\iota_X, d] = \iota_X d - (-1)^{(1)}(-1)^d \iota_X = \iota_X d + dt_X \]

(2) \( \omega : V \rightarrow V^*, \omega^* = -\omega \) If \( \omega \) is an isomorphism, then for any \( X \in V \) we have \( \omega(X, X) = 0 \), so that

\[ X \in X^\omega = \text{Ker} \omega(X) = \omega^{-1} \text{Ann} X \]

Thus, we have an isomorphism \( \omega^* : X^\omega/\langle X \rangle \xrightarrow{\cong} \text{Ann} X/\langle \omega X \rangle \) and

\[ \frac{\text{Ann} X}{\langle \omega X \rangle} = \frac{\langle X \rangle^*}{(X^\omega)^*} = \left( \frac{X^\omega}{\langle X \rangle} \right)^* \]

Then using induction, we can prove that \( V \) must be even dimensional.

2.2 Symplectic Manifolds

(continues the previous lecture)

For a manifold \( M \), consider its cotangent bundle \( T^*M \) equipped with the 2-form \( \omega = d\theta \), where \( \theta \in \Omega^1(T^*M) \) is such that \( \theta_v(v) = \alpha(\pi_*(v)) \). In coordinates \( (x^1, \ldots, x^n, a_1, \ldots, a_n) \), we have \( \theta = \sum_i a_i dx^i \) and therefore \( d\theta = \sum_i da_i \wedge dx^i \), as in the Darboux theorem. Thus, \( T^*M \) is symplectic.

**Definition 6.** A subspace \( W \) of a symplectic 2\( n \)-dimensional vector space \( (V, \omega) \) is called **isotropic** if \( \omega|_W = 0 \).

\( W \) is called **coisotropic** if its \( \omega \)-perpendicular subspace \( W^\omega \) is isotropic.

\( W \) is called **Lagrangian** if it is both isotropic and coisotropic.

Continued on next page...
There exist isotropic subspaces of any dimension $0,1,\ldots,n$, and coisotropic subspaces of any dimension $n,n+1,\ldots,2n$. Hence, Lagrangian subspaces must be of dimension $n$.

We have analogous definitions for submanifolds of a symplectic manifold $(M,\omega)$:

**Definition 7.** $L \xrightarrow{\mathcal{J}} (M,\omega)$ is called isotropic if $f^*\omega = 0$. When $\dim(L) = n$ it is called Lagrangian.

The graph of $0 \in C^\infty(M,T^*M)$, which is the zero section of $T^*M$, is Lagrangian.

More generally, $\Gamma_\xi$, the graph of $\xi \in C^\infty(M,T^*M)$ is a Lagrangian submanifold of $T^*M$ if and only if $d\xi = 0$. It is in this sense that we say that Lagrangian submanifolds of $T^*M$ are like generalized functions: $\{f\in C^\infty(M)\text{ gives rise to } df\}$, which is a closed 1-form, so $\Gamma_{df} \subset T^*M$ is Lagrangian.

**Proposition 1.** Suppose we have a diffeomorphism between two symplectic manifolds, $\varphi : (M_0,\omega_0) \rightarrow (M_1,\omega_1)$ and let $\pi_i : M_0 \times M_1 \rightarrow M_i$, $i = 0,1$ be the projection maps.

Then, $\text{Graph}(\varphi) \subset (M_0 \times M_1,\pi_0^*\omega_0 - \pi_1^*\omega_1)$ is Lagrangian if and only if $\varphi$ is a symplectomorphism.

### 2.3 Poisson geometry

**Definition 8.** A Poisson structure on a manifold $M$ is a section $\pi \in C^\infty(\wedge^2(TM))$ such that $[\pi,\pi] = 0$, where $[\cdot,\cdot]$ is the Shouten bracket.

**Remark.** $[\pi,\pi] \in C^\infty(\wedge^3(TM))$, so for a surface $\Sigma^{(2)}$, all $\pi \in C^\infty(\wedge^2(TM))$ are Poisson.

This defines a bracket on functions, called the Poisson bracket:

**Definition 9.** The Poisson bracket of two functions $f,g \in C^\infty(\wedge^0(TM))$ is

$$\{f,g\} = \pi(df,dg) = \iota(df \wedge dg) = [\pi,f],g$$

**Proposition 2.** The triple $(C^\infty(M),\{\cdot,\cdot\})$ is a Poisson algebra, i.e., it satisfies the properties below. For $f,g,h \in C^\infty(\wedge^0(TM))$,

- Leibniz rule: $\{f,gh\} = \{f,g\} h + g \{f,h\}$
- Jacobi identity: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

**Problem.** Write $\{f,g\}$ in coordinates for $\pi = \pi^i\partial_i \wedge \partial^j$.

A basic example of a Poisson structure is given by $\omega^{-1}$, where $\omega$ is a symplectic form on $M$, since

$$[\omega^{-1},\omega^{-1}] = 0 \iff d\omega = 0 \quad (6)$$

**Problem.** Prove (6) by testing $d\omega(X_f,X_g,X_h)$, for $f,g,h \in C^\infty(M)$.

Poisson manifolds are of interest in physics: given a function $H \in C^\infty(M)$ on a Poisson manifold $(M,\pi)$, we get a unique vector field $X_H = \pi(dH)$ and its flow $F^t_{X_H}$. $H$ is called Hamiltonian, and we usually think about it as energy.

We have $X_H(H) = \pi(dH,dH) = 0$, so $H$ is preserved by the flow. What other functions $f \in C^\infty(M)$ are preserved by the flow? A function $f \in C^\infty(M)$ is conserved by the flow if and only if $X_H(f) = 0$, equivalently $\{H,f\} = 0$, $f$ commutes with the Hamiltonian.

If we can find $k$ conserved quantities, $H_0 = H, H_1, H_2, \ldots, H_k$ such that $\{H_0,H_i\} = 0$, then the system must remain on a level surface $Z = \{x : (H_0,\ldots,H_k) = \ell\}$ for all time. Moreover, if $\{H_i,H_j\}$ for all $i,j$ then we get commutative flows $F^t_{X_{H_i}}$. If $Z$ is compact, connected, and $k = n$, then $Z$ is a torus $\mathbb{T}^n$, and the trajectory is a straight line in these coordinates. Also, $\mathbb{T}^n$ is Lagrangian.
**Problem.** Describe the Hamiltonian flow on $T^*M$ for $H = \pi^*f$, with $f \in C^\infty(M)$ and $\pi : T^*M \to M$. Show that a coordinate patch for $M$ gives a natural system of $n$ commuting Hamiltonians.

Let us now think about a Poisson structure, $\pi : T^* \to T$ and consider $\Delta = \text{Im}\pi$. $\Delta$ is spanned at each point $x$ by $\pi(df) = X_f$, Hamiltonian vector fields. The Poisson tensor is always preserved:

$$L_{X_f} \pi = [\pi, X_f] = [\pi, [\pi, f]] = (\pi, f) + (-1)^{1+1} [\pi, f] = -[\pi, f]$$

$$\implies L_{X_f} \pi = 0$$

If $\Delta_{x_0} = \langle X_{f_1}, \ldots, X_{f_k} \rangle$, then $Fl_{X_{f_1}}^{\pi_1} \circ \ldots \circ Fl_{X_{f_k}}^{\pi_k}(x_0)$ sweeps out $S \ni x_0$ submanifold of $M$ such that $TS = \Delta$.

**Example (of a generalized Poisson structure).** Let $M = \mathfrak{g}^*$, for $\mathfrak{g}$ a Lie algebra, $[\cdot, \cdot] \in \wedge^2 \mathfrak{g} \otimes \mathfrak{g}$. Then $TM = M \times \mathfrak{g}^*$ and $T^*M = M \times \mathfrak{g}$, and also $\wedge^2(TM) = M \times \wedge^2 \mathfrak{g}$, so $[\cdot, \cdot] \in C^\infty(\wedge^2 T\mathfrak{g}^*)$.

Given $f_1, f_2 \in C^\infty(M)$, their Poisson bracket is given by $\{f_1, f_2\}(x) = \langle df_1, df_2 \rangle, x\rangle$.

For $f, g \in \mathfrak{g}$ linear functions on $M$, we have

$$X_f(g) = \langle [f, g], x \rangle = \langle \text{ad}_f g, x \rangle = \langle g, -\text{ad}_f^* x \rangle$$

Thus $X_f = -\text{ad}_f^*$, so the the leaves of $\Delta = \text{Im}\pi$ are coadjoint orbits. If $S$ is a leaf, then

$$0 \longrightarrow N^*_S \longrightarrow T^*|_S^\pi \longrightarrow T|_S \longrightarrow 0$$

is a short exact sequence and we have an isomorphism $\pi_* : T^*S = \frac{T|_S^\pi}{N^*_S} \cong TS$, which implies that the leaf $S$ is symplectic.

Given $f, g \in C^\infty(S)$, we can extend them to $\tilde{f}, \tilde{g} \in C^\infty(M)$. The Poisson bracket $\{\tilde{f}, \tilde{g}\}_\pi$ is independent of choice of $\tilde{f}, \tilde{g}$, so $\{f, g\}_\pi = \{\tilde{f}, \tilde{g}\}_\pi$ is well defined.

Therefore, giving a Poisson structure on a manifold is the same as giving a “generalized” foliation with symplectic leaves.

When $\pi$ is Poisson, $[\pi, \pi] = 0$, we can define

$$d_\pi = [\pi, \cdot] : C^\infty(\wedge^k T) \to C^\infty(\wedge^{k+1} T)$$

Note that $[\pi, \cdot]$ is of degree $(2-1)$, so it makes sense to call it $d_\pi$. Also,

$$d_\pi^2(A) = [\pi, [\pi, A]] = [[\pi, \pi], A] - [\pi, [\pi, A]] = -[\pi, [\pi, A]]$$

$$\implies d_\pi^2 = 0$$

Thus, we have a chain complex

$$\ldots \longrightarrow C^\infty(\wedge^{k-1} T) \xrightarrow{d_\pi} C^\infty(\wedge^{k} T) \xrightarrow{d_\pi} C^\infty(\wedge^{k+1} T) \longrightarrow \ldots$$

Moreover, if $m_f$ denotes multiplication by $f \in C^\infty(M)$,

$$[d_\pi, m_f] \psi = d_\pi (f \psi) - f d_\pi \psi = [\pi, f \psi] - f [\pi, \psi] = [\pi, f] \wedge \psi = \iota_{df} \pi \wedge \psi$$

But for any $\xi \in T^*$, $\xi \neq 0$, $(\xi \pi)\wedge : \wedge^k T \to \wedge^{k+1} T$ is exact only for $\iota_{\xi} \pi \neq 0$. So, if $\pi$ is not invertible, $d_\pi$ is not an elliptic complex, and the Poisson cohomology groups, $H^k_\pi(M) = \text{Ker} d_\pi|_{\wedge^k T} / \text{Im} d_{\pi+1}|_{\wedge^{k+1} T}$ could be infinite dimensional on a compact $M$.

Let us look at the first such groups:

- $H^0_\pi(M) = \{ f : d_\pi f = 0 \} = \{ f : X_f = 0 \} = \{ \text{Casimir functions, i.e. functions s.t. } f(g) = 0 \text{ for all } g \}$
- $H^1_\pi(M) = \{ X : d_\pi X = 0 \} / \text{Im} d_{\pi+1} = \{ \text{infinitesimal symmetries of Poisson vector fields} \} / \text{Hamiltonians}$
- $H^2_\pi(M) = \{ P \in C^\infty(\wedge^2 T) : [\pi, P] = 0 \}$ is tangent space to the moduli space of Poisson structures