9 Lecture 9 (Notes: K. Venkatram)

Last time, we talked about the geometry of a connected Lie group $G$. Specifically, for any $a$ in the corresponding Lie algebra $\mathfrak{g}$, one can define $a^L\big|_g = L_a a$ and choose $\theta^L \in \Omega^1(G, \mathfrak{g})$ s.t. $\theta^L(a^L) = a$. For instance, for $\text{GL}_n$, with coordinates $g = [g_{ij}]$, one has $\theta^L = g^{-1}dg$, and similarly $\theta^R = dg^{-1}$. This implies that $dg \wedge \theta^L + gd\theta^L = 0 \Rightarrow d\theta^L + \theta^L \wedge \theta^L = 0 \Rightarrow d\theta^L + \frac{1}{2}[\theta^L, \theta^L] = 0$, the latter of which is the Maurer-Cartan equation.

**Problem.**
1. Extend this proof so that it works in the general case.
2. Show $j^* \theta^R = -\theta^L$.
3. Show $d\theta^R - \frac{1}{2}[\theta^R, \theta^R] = 0$.
4. Show $\theta^R(a^L)\big|_g = \text{Ad}_g a$ for $a \in \mathfrak{g}, g \in G$.

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9.1 Bilinear forms on groups

Let $G$ be a connected real Lie group, $B$ a symmetric nondegenerate bilinear form on $g$. This extends to a left-invariant metric on $G$, and $B$ is invariant under right translation
\[ \Leftrightarrow B([X, Y], Z) + B(Y, [X, Z]) = 0 \quad \forall X, Y, Z. \]
If this is true, we obtain a bi-invariant (pseudo-Riemannian) metric on $G$.

**Remark.** Geodesics through $e$ are one-parameter subgroups $\Leftrightarrow B$ is bi-invariant. See Helgason for Riemannian geometry of Lie groups and homogeneous spaces.

**Example.** Let $B$ be the Killing form on a semisimple Lie group, i.e. $B(a, b) = \text{Tr}_g(\text{ad}_a \text{ad}_b)$ for $\mathfrak{s}|_m, \mathfrak{s} \circ \mathfrak{m}, \mathfrak{sp}_m$ a constant multiple of $\text{Tr}(X, Y)$. Now, we can form the Cartan 3-form
\[
H = \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]) = \frac{1}{12} B(\theta^R, [\theta^R, \theta^R])
\] (7)
This $H$ is bi-invariant, and thus closed. When $G$ is simple, compact, and simply connected, the Killing form gives $\lambda[H]$ as a generator for $H^3(G, \mathbb{Z}) = \mathbb{Z}$. (See Brylinski.) For instance, given $\mathfrak{g} = \mathfrak{s}|_n, \theta^L = g^{-1}dg$, one has $H = \text{Tr}(g^{-1}dg)^3$.

9.1.1 Key calculation

Let $m, p_1, p_2 : G \times G \to G$ be the multiplication and projection maps respectively. Then
\[
m^*H = \text{Tr}((gh)^{-1}d(gh))^3 = \text{Tr}(h^{-1}g^{-1}(dh + dg)h)^3
\]
\[= \text{Tr}(h^{-1}g^{-1}) + \text{Tr}(g^{-1}d) + \text{Tr}(dhh^{-1}g^{-1}dg) + \text{Tr}(dh^{-1}g^{-1}dg)^2)\] (8)
Now, define $\theta = dh^{-1}, \Omega = g^{-1}dg$, so $d\theta = \theta \land \theta$ and $d\Omega = -\Omega \land \Omega$. Then
\[d\text{Tr}(dh^{-1}g^{-1}dg) = d\text{Tr}(\theta \land \Omega) = d\text{Tr}(\theta \land \Omega - \theta \land d\Omega)
\]
\[= \text{Tr}(\theta \land \Omega + \theta \land \Omega \land \Omega)\] (9)
So, $m^*H - p_1^*H - p_2^*H = dr$, where $r = \text{Tr}(dh^{-1}g^{-1}dg) = B(p_1^*\theta^L, p_2^*\theta^R) \in \Omega^2(G \times G)$. Now, recall that given a metric $g : V \to V^*$, we have a decomposition $V \oplus V^* = C_+ \oplus C_-$ for $C_\pm = G_\pm$. Moreover, any Dirac structure $L \subset V \oplus V^*$ can be written as the graph of $A \in \text{O}(V, g)$ thought of as $A : C_+ \to C_-$. Now, for $X \in V$, let $X = X \pm gX \in C_\pm$. Then $L^A = \{ (X^+ \pm (AX)^-) | X \in V \}$ are the Dirac structures. Note that
\[
(X^+ \pm (AX)^-, X^+ \pm (AX)^-) = g(X, X) - g(AX, AX) = 0
\] (10)

Let $B$ be a bi-invariant metric on $G$. Then the map $A_x = L_{x^{-1}}, R_{x^*} : T_xG \to T_xG, a^L \mapsto a^R$ is orthogonal for $B$ and $\text{ad}(G)$-invariant, since
\[
T_xG \xrightarrow{A_x} T_xG \quad \text{ad}_g \bigg|_{T_xG} \xrightarrow{\text{ad}_g} T_{g^{-1}}T_xG \xrightarrow{A_{g^{-1}}^*} T_{g^{-1}}G
\] (11)
where $\text{ad}_g = L_g, R_{g^{-1}}$. Thus, we find that
\[
\text{ad}_g A_x \text{ad}_g^{-1} = L_g R_{g^{-1}} R_{x^{-1}} R_{g} L_{g^{-1}} = L_{g^{-1}x^{-1}} R_{g^{-1}} = A_{g^{-1}}
\] (12)

Overall, $L_\pm(A)$ are $\text{ad}(G)$-invariant almost Dirac structures in $(T \oplus T^*)(G)$. $T_xG$ is spanned by the $a^L$, so $L_+$ is spanned by $(a^L)^+ = a^L + B(a^L) + a^L - B(a^L)$ and $L_+ = \langle a^L + a^R + B(a^L - a^R) \rangle$. Recall that $\theta^L(a^L) = a$ so $\langle a^L + a^R + B(a^L - a^R) \rangle = \langle a^L + a^R + B(\theta^L - \theta^R, a) \rangle$. Similarly, $L_- = \langle a^L - a^R + B(\theta^L + \theta^R, a) \rangle$. 

28
Remark. Since $a^L - a^R$ generates the adjoint action, $[a^L - a^R, b^L - b^R] = [a, b]^L - [a, b]^R$. But $[a^L + a^R, b^L + b^R] = [a, b]^L + [a, b]^R$ is not integrable. $L_-(A)$ is integrable, however, w.r.t. the Courant bracket twisted by $H = B(\theta^L, [\theta^L, \theta^L])$. 