MIRROR SYMMETRY: LECTURE 3
DENIS AUROUX

Last time, we say that a deformation of \((X, J)\) is given by

\[
\{ s \in \Omega^{0,1}(X, TX) | \bar{\partial}s + \frac{1}{2}[s, s] = 0 \}/\text{Diff}(X)
\]

To first order, these are determined by \(\text{Def}_1(X, J) = H^1(X, TX)\), but extending these to higher order is obstructed by elements of \(H^2(X, TX)\). In the Calabi-Yau case, recall that:

**Theorem 1** (Bogomolov-Tian-Todorov). For \(X\) a compact Calabi-Yau \((\Omega^*_X \cong \mathcal{O}_X)\) with \(H^0(X, TX) = 0\) (automorphisms are discrete), deformations of \(X\) are unobstructed.

Note that, if \(X\) is a Calabi-Yau manifold, we have a natural isomorphism \(TX \cong \Omega^{n-1}_X\), \(v \mapsto i_v\Omega\), so

\[
H^0(X, TX) = H^{n-1,0}(X) \cong H^{0,1}
\]

and similarly

\[
H^1(X, TX) = H^{n-1,1}, H^2(X, TX) = H^{n-1,2}
\]

1. **Hodge theory**

Given a Kähler metric, we have a Hodge \(*\) operator and \(L^2\)-adjoints

\[
d^* = - * d*, \bar{\partial}^* = - * \partial^*
\]

and Laplacians

\[
\Delta = dd^* + d^*d, \Box = \bar{\partial}\partial^* + \partial^*\partial
\]

Every \((d/\bar{\partial})\)-cohomology class contains a unique harmonic form, and one can show that \(\Box = \frac{1}{2}\Delta\). We obtain

\[
H^k_{\partial\bar{\partial}}(X, \mathbb{C}) \cong \text{Ker} \left( \Delta : \Omega^k(X, \mathbb{C}) \to \Omega^k(X, \mathbb{C}) \right) = \text{Ker} \left( \Box : \Omega^k \to \Omega^k \right)
\]

\[
\cong \bigoplus_{p+q=k} \text{Ker} \left( \Box : \Omega^{p,q} \to \Omega^{p,q} \right) \cong \bigoplus_{p+q=k} H^p_{\partial\bar{\partial}}(X)
\]
The Hodge * operator gives an isomorphism \( H^{p,q} \cong H^{n-p,n-q} \). Complex conjugation gives \( H^{p,q} \cong \overline{H^{q,p}} \), giving us a Hodge diamond

\[
\begin{array}{cccc}
h_{n,n} & h_{n-1,n} & \cdots & \cdots & h_{0,n} \\
h_{n,n-1} & h_{n-1,n-1} & \cdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
h_{n,0} & \cdots & \cdots & h_{1,0} & h_{0,0}
\end{array}
\]

(7)

For a Calabi-Yau, we have

\[
H^{p,0} \cong H^{n-n-p} = H^{n-p}_q(X, \Omega^n_X) \cong H^{n-p}_q(X, \mathcal{O}_X) = H^{0,n-p} \cong \overline{H^{n-p,0}}
\]

Specifically, for a Calabi-Yau 3-fold with \( h^{1,0} = 0 \), we have a reduced Hodge diamond

\[
\begin{array}{cccc}
1 & 0 & 0 & 1 \\
0 & h^{1,1} & h^{2,1} & 0 \\
0 & h^{2,1} & h^{1,1} & 0 \\
1 & 0 & 0 & 1
\end{array}
\]

(9)

Mirror symmetry says that there is another Calabi-Yau manifold whose Hodge diamond is the mirror image (or 90 degree rotation) of this one.

There is another interpretation of the Kodaira-Spencer map \( H^1(X, TX) \cong H^{n-1,1} \). For \( X = (X, J_t)_{t \in S} \) a family of complex deformations of \((X, J)\), \( c_1(K_X) = -c_1(TX) = 0 \) implies that \( \Omega^n_{(X,J_t)} \cong \mathcal{O}_X \) under the assumption \( H^1(X) = 0 \), so we don’t have to worry about deforming outside the Calabi-Yau case. Then \( \exists [\Omega_t] \in H^{n,0}_q(X) \subset H^n(X, \mathbb{C}). \) How does this depend on \( t \)? Given \( \frac{\partial}{\partial t} \in T_0 S, \frac{\partial}{\partial t} \in \Omega^{n,0} \oplus \Omega^{n-1,1} \) by Griffiths transversality:

\[
\alpha_t \in \Omega^{p,q}_{J_t} \implies \frac{\partial}{\partial t} \alpha_t \in \Omega^{p,q} + \Omega^{p-1,q+1} + \Omega^{p+1,q-1}
\]

(10)
Since $\frac{\partial \Omega_t}{\partial t} |_{t=0}$ is d-closed ($d\Omega_t = 0$), ($\frac{\partial \Omega_t}{\partial t} |_{t=0})^{(n-1,1)}$ is $\bar{\partial}$-closed, while
\begin{equation}
\bar{\partial}((\frac{\partial \Omega_t}{\partial t} |_{t=0})^{(n-1,1)}) + \bar{\partial}(\frac{\partial \Omega_t}{\partial t} |_{t=0})^{(n-1,1)} = 0
\end{equation}
Thus, $\exists (\frac{\partial \Omega_t}{\partial t} |_{t=0})^{(n-1,1)} \in H^{n-1,1}(X)$.

For fixed $\Omega_0$, this is independent of the choice of $\Omega_t$. If we rescale $f(t)\Omega_t$,
\begin{equation}
\frac{\partial}{\partial t}(f(t)\Omega_t) = \frac{\partial f}{\partial t}\Omega_t + f(t)\frac{\partial \Omega_t}{\partial t}
\end{equation}
Taking $t \to 0$, the former term is $(n, 0)$, while for the latter, $f(0)$ scales linearly with $\Omega^0$.
\begin{equation}
H^{n-1,1}(X) = H^1(X, \Omega_X^{n-1}) \cong H^1(X, TX)
\end{equation}
and the two maps $T_0S \to H^{n-1,1}(X), H^1(X, TX)$ agree. Hence, for $\theta \in H^1(X, TX)$
a first-order deformation of complex structure, $\theta \cdot \Omega \in H^1(X, \Omega_X^{n-1} \otimes TX) =
H^{n-1,1}(X)$ and (the Gauss-Manin connection) $[\nabla_\theta \Omega]^{(n-1,1)} \in H^{n-1,1}(X)$ are the
same. We can iterate this to the third-order derivative: on a Calabi-Yau threefold, we have
\begin{equation}
\langle \theta_1, \theta_2, \theta_3 \rangle = \int_X \Omega \wedge (\theta_1 \cdot \theta_2 \cdot \theta_3 \cdot \Omega) = \int_X \Omega \wedge (\nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega)
\end{equation}
where the latter wedge is of a $(3, 0)$ and a $(0, 3)$ form.

2. PSEUDOHOLOMORPHIC CURVES

(reference: McDuff-Salamon) Let $(X^{2n}, \omega)$ be a symplectic manifold, $J$ a compatible
almost-complex structure, $\omega(\cdot, J\cdot)$ the associated Riemannian metric.
Furthermore, let $(\Sigma, j)$ be a Riemann surface of genus $g, z_1, \ldots, z_k \in \Sigma$ market
points. There is a well-defined moduli space $\mathcal{M}_{g,k} = \{(\Sigma, j, z_1, \ldots, z_k)\}$ modulo
biholomorphisms of complex dimension $3g - 3 + k$ (note that $\mathcal{M}_{0,3} = \{pt\}$).

**Definition 1.** $u : \Sigma \to X$ is a $J$-holomorphic map if $J \circ du = du \circ J$, i.e.
$\bar{\partial}_J u = \frac{1}{2}(du + Jduj) = 0$. For $\beta \in H_2(X, \mathbb{Z})$, we obtain an associated moduli space
\begin{equation}
M_{g,k}(X, J, \beta) = \{(\Sigma, j, z_1, \ldots, z_k), u : \Sigma \to X | u_*[\Sigma] = \beta, \bar{\partial}_J u = 0\} / \sim
\end{equation}
where $\sim$ is the equivalence given by $\phi$ below.
\begin{equation}
\begin{array}{ccc}
\Sigma, z_1, \ldots, z_k & \overset{u}{\longrightarrow} & X \\
\phi |_{\Sigma'} & \cong & u' \\
\Sigma', z_1', \ldots, z_k' & & \\
\end{array}
\end{equation}
This space is the zero set of the section $\bar{\partial}_J$ of $\mathcal{E} \to Map(\Sigma, X)_\beta \times \mathcal{M}_{g,k}$, where $\mathcal{E}$
is the (Banach) bundle defined by $\mathcal{E}_u = W^{r,p}(\Sigma, \Omega^0_{\Sigma} \otimes u^*TX)$. 
We can define a linearized operator

\[ D_\overline{\beta} : W^{r+1,p}(\Sigma, u^*TX) \times TM_{g,k} \to W^{r,p}(\Sigma, \Omega^0_{\Sigma} \otimes U^*TX) \]

\[ D_\overline{\beta}(v, j') = \frac{1}{2}(\nabla v + J\nabla v_j + (\nabla_v J) \cdot du \cdot j + J \cdot du \cdot j') \]

(17) \[ = \overline{\partial} v + \frac{1}{2}(\nabla_v J)du \cdot j + \frac{1}{2}J \cdot du \cdot j' \]

This operator is Fredholm, with real index

\[ \text{index}_\mathbb{R} D_\overline{\beta} := 2d = 2(c_1(TX), \beta) + n(2 - 2g) + (6g - 6 + 2k) \]

One can ask about transversality, i.e. whether we can ensure that \( D_\overline{\beta} \) is onto at every solution. We say that \( u \) is regular if this is true at \( u \): if so, \( \mathcal{M}_{g,k}(X, J, \beta) \) is smooth of dimension \( 2d \).

**Definition 2.** We say that a map \( \Sigma \to X \) is simple (or “somewhere injective”) if \( \exists z \in \Sigma \) s.t. \( du(z) \neq 0 \) and \( u^{-1}(u(z)) = \{z\} \).

Note that otherwise \( u \) will factor through a covering \( \Sigma \to \Sigma' \). We set \( \mathcal{M}_{g,k}^*(X, J, \beta) \) to be the moduli space of such simple curves.

**Theorem 2.** Let \( \mathcal{J}(X, \omega) \) be the set of compatible almost-complex structures on \( X \): then

\[ \mathcal{J}^{reg}(X, \beta) = \{ J \in \mathcal{J}(X, \omega) | \text{ every simple } J\text{-holomorphic curve in class } \beta \text{ is regular} \} \]

is a Baire subset in \( \mathcal{J}(X, \omega) \), and for \( J \in \mathcal{J}^{reg}(X, \beta) \), \( \mathcal{M}_{g,k}^*(X, J, \beta) \) is smooth (as an orbifold, if \( \mathcal{M}_{g,k} \) is an orbifold) of real dimension \( 2d \) and carries a natural orientation.

The main idea here is to view \( \overline{\partial}_Ju = 0 \) as an equation on \( \text{Map}(\Sigma, X) \times \mathcal{M}_{g,k} \times \mathcal{J}(X, \omega) \ni (u, j, J). \) Then \( D_\overline{\beta} \) is easily seen to be surjective for simple maps. We have a “universal moduli space” \( \mathcal{M}^* \xrightarrow{\pi_J} \mathcal{J}(X, \omega) \) given by a Fredholm map, and by Sard-Smale, a generic \( J \) is a regular value of \( \pi_J. \) This universal moduli space is \( \mathcal{M}^* = \bigsqcup_{J \in \mathcal{J}(X, \omega)} \mathcal{M}^*_{g,k}(X, J, \beta). \) For such \( J \), \( \mathcal{M}^*_{g,k}(X, J, \beta) \) is smooth of dimension \( 2d \), and the tangent space is \( \text{Ker} (D_\overline{\beta}). \) For the orientability, we need an orientation on \( \text{Ker} (D_\overline{\beta}). \) If \( J \) is integrable, the \( D_\overline{\beta} \) is \( \mathbb{C} \)-linear \( (D_\overline{\beta} = \overline{\partial}), \) so \( \text{Ker} \) is a \( \mathbb{C} \)-vector space. Moreover, \( \forall J_0, J_1 \in \mathcal{J}^{reg}(X, \beta) \), \( \exists \) a (dense set of choices of) path \( \{J_t\}_{t \in [0,1]} \) s.t. \( \bigcup_{t \in [0,1]} \mathcal{M}^*_{g,k}(X, J_t, \beta) \) is a smooth oriented cobordism. We still need compactness.