1. Gromov-Witten Invariants

Recall that if \((X, \omega)\) is a symplectic manifold, \(J\) an almost-complex structure, \(\beta \in H_2(X, \mathbb{Z})\), \(\overline{M}_{g,k}(X, J, \beta)\) is the set of (possibly nodal) \(J\)-holomorphic maps to \(X\) of genus \(g\) representing class \(\beta\) with \(k\) marked points up to equivalence. This is not a nice moduli space, but does have a fundamental class \([\overline{M}_{g,k}(X, J, \beta)] \in H_{2d}(\overline{M}_{g,k}(X, J, \beta), \mathbb{Q})\), where \(2d = c_1(TX), \beta + 2(n - 3)(1 - g) + 2k\). We further have an evaluation map \(ev = (ev_1, \ldots, ev_n) : \overline{M}_{g,k}(X, J, \beta) \to X^k, (\Sigma, z_1, \ldots, z_k, u) \mapsto (u(z_1), \ldots, u(z_k))\). Then the Gromov-Witten invariants are defined for \(\alpha_1, \ldots, \alpha_k \in H^*(X)\), \(\sum \deg \alpha_i = 2d\) by

\[
\langle \alpha_1, \ldots, \alpha_k \rangle_{g, \beta} = \int_{[\overline{M}_{g,k}(X, J, \beta)]} ev^*_1 \alpha_1 \wedge \cdots \wedge ev^*_k \alpha_k \in \mathbb{Q}
\]

Or dually, for \(\alpha_i = PD(C_i)\), \(\#(ev_*[\overline{M}_{g,k}(X, J, \beta)] \cap (C_1 \times \cdots \times C_k)) \in \mathbb{Q}\).

For a Calabi-Yau 3-fold, we're interested in \(g = 0, k = 3\), so \(\Sigma = (S^2, \{0, 1, \infty\})\). For \(\deg \alpha_i = 2\), \(\alpha_i = PD(C_i)\), \(C_i\) cycles transverse to the evaluation map, we have

\[
\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0, \beta} = \#\{u : S^2 \to X J\text{-hol. of class }\beta, u(0) \in C_1, u(1) \in C_2, u(\infty) \in C_3\} / \sim
\]

Reparameterization acts transitively on triples of points, so

\[
\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0, \beta} = (C_1 \cdot \beta)(C_2 \cdot \beta)(C_3 \cdot \beta) \#\{u : S^2 \to X J\text{-hol. of class }\beta\} / \sim = (\int_\beta \alpha_1)(\int_\beta \alpha_2)(\int_\beta \alpha_3) \cdot \#(\overline{M}_{0,0}(X, J, \beta))
\]

We denote by \(N_\beta \in \mathbb{Q}\) the latter number \(\#(\overline{M}_{0,0}(X, J, \beta))\). This works when \(\beta \neq 0\): when \(\beta = 0\), we instead obtain

\[
\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0, 0} = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3
\]
1.1. Yukawa coupling. Physicists write this as

\[
\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \sum_{0 \neq \beta \in H_2(X,\mathbb{Z})} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta} e^{2\pi i \int_B B + i \omega}
\]

We want to ignore issues of convergence, and so treat this is a formal power series

\[
\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \sum_{\beta \neq 0} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta} q^\beta \in \Lambda
\]

where \( \Lambda \) is the completion of the group ring \( \mathbb{Q}[H_2(X,\mathbb{Z})] = \{ \sum a_i q^{\beta_i} | a_i \in \mathbb{Q}, \beta_i \in H_2 \} \). Specifically, we allow infinite sums provided that \( \int_{\beta_i} \omega \to +\infty \).

1.2. Quantum cohomology. This is new product structure on \( H^*(X) \) deformed by this coupling. Namely, pick a basis \( (\eta_i) \) of \( H^*(X) \), \( (\eta^i) \) the dual basis, i.e. \( \int_X \eta_i \wedge \eta^j = \delta_{ij} \). Set

\[
a_1 \ast a_2 = \sum_i \langle \alpha_1, \alpha_2, \eta^i \rangle \eta_i = \alpha_1 \wedge \alpha_2 + \sum_{\beta \neq 0} \langle \alpha_1, \alpha_2, \eta^i \rangle_{0,\beta} q^\beta \eta_i
\]

Definition 1. The quantum cohomology of \( X \) is \( QH^*(X) = (H^*(X;\Lambda), \ast) \).

Theorem 1. This is an associative algebra.

The proof of this relies on understanding the relationship between 4 point GW invariants and various 3 point ones.

1.3. Kähler moduli. We can view \( q \) as the coordinates on a Kähler moduli space: for \( (X, J) \)-complex, the Kähler cone \( \mathcal{K}(X, J) = \{ [\omega] | \omega \text{ Kähler} \} \subset H^{1,1}(X) \cap H^2(X,\mathbb{R}) \) is a open, convex cone. Its real dimension is \( h^{1,1}(X) \), and we can make it a complex manifold by adding the “B-field”.

Definition 2. Let \( (X, J) \) be a Calabi-Yau 3-fold with \( h^{1,0} = 0 \) (so \( h^{2,0} = 0 \) and \( H^{1,1} = H^2 \)). Then the complexified Kähler moduli space is

\[
\mathcal{M}_{Kah} = (H^2(X,\mathbb{R}) + i\mathcal{K}(X, J))/H^2(X,\mathbb{Z})
\]

\[
= \{ [B + i\omega], \omega \text{ Kähler} \}/H^2(X,\mathbb{Z})
\]

Choose a basis \( (e_i) \) of \( H^2(X,\mathbb{Z}) \), \( e_1, \ldots, e_m \in \mathcal{K}(X, J) \) (which exists by openness). We can write \( [B + i\omega] = \sum t_i e_i, t_i \in \mathbb{C}/\mathbb{Z} \), so we have coordinates on \( \mathcal{M}_{Kah} \) given by \( q_i = \exp(2\pi i t_i) \). Thus, \( \mathcal{M}_{Kah} \) is an open subset of \( (\mathbb{C}^*)^m \) which contains \( (\mathbb{D}^*)^m \), where \( \mathbb{D}^* = \{ q | 0 < |q| < 1 \} \).

We now can associate \( q^\beta \) to \( q_1^{d_1} \cdots q_m^{d_m} \), where \( d_i = \int_B e_i \) for \( e_i \geq 0 \) integers (it is an integer cohomology class integrated against an integer homology class): explicitly, \( q_1^{d_1} \cdots q_m^{d_m} = \exp(2\pi i \sum d_i t_i) = \exp(2\pi i \int_B B + i \omega) \). We can view \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle \) as a power series in the \( q_i \), though we still do not know about convergence.
1.4. Gromov-Witten invariants vs. numbers of curves. We have, for \( \alpha_1, \alpha_2, \alpha_3 \in H^2(X) \),

\[
\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \sum_{\beta \neq 0} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0, \beta} q^\beta
\]

(9)

\[
= \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \sum_{\beta \neq 0} (\int_\beta \alpha_1)(\int_\beta \alpha_2)(\int_\beta \alpha_3) N_\beta q^\beta
\]

This is much like our formula from the first class, except the latter term had the form \( n_\beta \frac{q^\beta}{1-q^\beta} \) and \( n_\beta \) as the number of “rational curves of class \( \beta \)”. The discrepancy comes from the existence of multiple covers. Let \( C \subset X \) be an embedded rational curve in a Calabi-Yau 3-fold. A theorem of Grothendieck says that a holomorphic bundle over \( \mathbb{P}^1 \) splits as \( \bigoplus \mathcal{O}_d \), where \( \mathcal{O}(d) \) is the sheaf whose sections are homogeneous degree \( d \) holomorphic functions on \( \mathbb{C}^2 \) and \( \mathcal{O}(-1) \) is the tautological bundle. Writing \( NC \cong \mathcal{O}_d \oplus \mathcal{O}_d \), we obtain

\[
0 = c_1(TX)[C] = c_1(NC)[C] + c_1(TC)[C] = d_1 + d_2 + 2
\]

so \( d_1 + d_2 = -2 \). The “generic case” is \( d_1 = d_2 = -1 \), in which case \( C \) is automatically regular as a \( J \)-holomorphic curve. The contribution of \( C \) to the Gromov-Witten invariant \( N_{[C]} \) is precisely 1. On the other hand, there is a component \( \mathcal{M}(kC) \subset \mathcal{M}_{0,0}(X, J, k[C]) \) consisting of \( k \)-fold covers of \( C \). What is \( \# [\mathcal{M}(kC)] \)?

**Theorem 2.** If \( NC \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \), then the contribution of \( C \) to \( N_{[k[C]]} \) is \( \frac{1}{k^3} \).

There are various proofs, all of which are somewhat difficult. For instance, Voisin shows that \( \exists \) perturbed \( \bar{\partial} \)-equations \( \bar{\partial} J u = \nu(z, u(z)) \) s.t. the moduli space \( \overline{\mathcal{M}}_3(kC) \) (of perturbed \( J \)-holomorphic maps with 3 marked points representing \( k[C] \) and whose image lies in a neighborhood of \( C \)) is smooth and has real dimension 6. Moreover, \( (ev_1 \times ev_2 \times ev_3)_*[\overline{\mathcal{N}}_3(kC)] = [C \times C \times C] \in H_6(X \times X \times X) \). Then the contribution of \( C \) to \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,[k[C]]} \) is

\[
\int_{ev_*[\overline{\mathcal{N}}_3]} \alpha_1 \times \alpha_2 \times \alpha_3 = (\int_C \alpha_1)(\int_C \alpha_2)(\int_C \alpha_3) = \frac{1}{k^3}(\int_{kC} \alpha_1)(\int_{kC} \alpha_2)(\int_{kC} \alpha_3)
\]

We expect that (\*\*) \( N_\beta = \sum_{\beta = k\gamma} \frac{1}{k^3} n_\gamma \).

**Remark.** We do not know if \( n_\gamma \) is what we think it is, but we use this formula as a definition; see the Gopakumar-Vafa conjecture, which claims that \( n_\gamma \in \mathbb{Z} \), and the theory of Donaldson-Thomas invariants and MNOP conjectures.
Assuming (*), we have
\[ \sum_{\beta}(\int_{\beta} \alpha_1)(\int_{\beta} \alpha_2)(\int_{\beta} \alpha_3)N_{\beta}q^\beta = \sum_{k\gamma}(\int_{k\gamma} \alpha_1)(\int_{k\gamma} \alpha_2)(\int_{k\gamma} \alpha_3)\frac{n_\gamma}{k^3}q^{k\gamma} \]
(12)

\[ = \sum_{\gamma}(\int_{\gamma} \alpha_1)(\int_{\gamma} \alpha_2)(\int_{\gamma} \alpha_3)n_\gamma \sum_{k \geq 1} k^{k\gamma} \]

Where we are headed: we correspond this pairing to
(13) \[ \langle \theta_1, \theta_2, \theta_3 \rangle = \int_X \Omega \wedge (\nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega) \]
on \[ H^{2,1}(\tilde{X}). \]