18.969 Topics in Geometry: Mirror Symmetry
Spring 2009

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0.1. Lagrangian Floer Homology (contd). Let \((M, \omega)\) be a symplectic manifold, \(L_0, L_1\) compact Lagrangian submanifolds. Formally, Floer homology is Morse theory for the action functional on the path space \(\mathcal{P}(L_0, L_1)\), which has as critical points the constant paths. More precisely, the actual functional is a map \(\tilde{A} : \mathcal{P}(L_0, L_1) \to \mathbb{R}\), where \(\mathcal{P}(L_0, L_1)\) is the universal cover of the path space, i.e. pairs \((\gamma, [u])\) where \(\gamma\) is a path between \(L_0\) and \(L_1\) and \([u]\) is a homotopy between \(\gamma\) and some fixed base path \(*\). Then \(A(\gamma, [u]) = \int u^* \omega\), and for \(v\) a vector field along \(\gamma\),

\[
dA(\gamma) \cdot v = \int_{[0,1]} \omega(\dot{\gamma}, v) dt = \int_{[0,1]} g(J\gamma; v) dt = \langle J\dot{\gamma}, v \rangle_{L^2}
\]

The critical points are constant paths \(\dot{\gamma} = 0\), and the gradient flow lines are \(J\)-holomorphic curves \(\frac{\partial v}{\partial s} = -J\dot{\gamma}\).

However, no one has managed to run this Morse theory rigorously. The actual setup assumes \(L_0, L_1\) are transverse, and as before, define the Novikov ring as 
\[
\Lambda = \{ \sum a_i T^{\lambda_i} \mid \lambda_i \to \infty \}
\]
and the Floer complex \(CF(L_0, L_1)\) as the free \(\Lambda\)-module \(\Lambda^{[L_0 \cap L_1]}\) generated by \(L_0 \cap L_1\). We look at \(u : \mathbb{R} \times [0,1] \to M\) equipped with a compatible almost-complex structure \(J\) s.t.

- \(\partial J u = 0, \text{ or } \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = 0\).
- \(u(s,0) \in L_0, u(s,1) \in L_1\).
- \(\lim_{s \to +\infty} u(s, t) = p, \lim_{s \to -\infty} u(s, t) = q\) for \(\{p,q\} \subset L_0 \cap L_1\).
- \(E(u) = \int u^* \omega = \int_{\mathbb{R} \times [0,1]} \left| \frac{\partial u}{\partial s} \right|^2 ds dt < \infty\).

We consider the space of solutions \(\mathcal{M}(p, q, [u], J)\) for fixed \(p, q \in L_0 \cap L_1, [u]\) a homotopy class as above, and \(J\) a given almost-complex structure. The above problem is a Fredholm problem, and the expected dimension of \(\mathcal{M} = \text{ind}(\partial J)\) is called the Maslov index. The Maslov index comes from \(\pi_1(\Lambda \text{Gr}) = \mathbb{Z}\). Explicitly, let \(L_0, L_1(t)_{t \in [0,1]}\) be Lagrangian subspaces of \(\mathbb{R}^{2n}\) s.t. \(L_1(0), L_1(1)\) intersect \(L_0\) transversely. The Maslov index of \((L_1(t); L_0)\) is the number of times that \(L_1(t)\) is non-transverse to \(L_0\) with multiplicities and signs. For instance, for \(L_0 = \mathbb{R}^n \subset \mathbb{C}^n\), \(L_1(t) = (e^{i\theta_1(t)} \mathbb{R}) \times \cdots \times (e^{i\theta_n(t)} \mathbb{R})\) with all \(\theta_i\)'s increasing past 0, the Maslov index is \(n\). In general, given a homotopy \(u\), we can trivialize \(u^*TM\), and \(u^*|_{\mathbb{R} \times 0}(TL_0), u^*|_{\mathbb{R} \times 1}(TL_1)\) are 2 paths of Lagrangian subspaces. We can trivialize
so that $TL_0$ remains constant, and $\text{ind}(u)$ is the Maslov index of the path $TL_1$ relative to $TL_0$ as one goes from $p$ to $q$.

Now, we want to define

$$\partial(p) = \sum_{q \in L_0 \cap L_1} \#(\mathcal{M}(p, q, \phi, J)/\mathbb{R})T^\omega(\phi) \cdot q$$

(2)

The issues that arise are: transversality, compactness and bubbling, the orientation of $\mathcal{M}$, and whether $\partial^2 = 0$.

**Theorem 1.** If $[\omega] \cdot \pi_2(M) = 0$ and $[\omega] \cdot \pi_2(M, L_i) = 0$, then $\partial$ is well-defined, $\partial^2 = 0$, and $HF(L_0, L_1) = H^*(CF, \partial)$ is independent of the chosen $J$ and invariant under Hamiltonian isotopies of $L_0$ and/or $L_1$.

**Corollary 1.** If $[\omega] \cdot \pi_2(M, L) = 0$ and $\psi$ is a Hamiltonian diffeomorphism s.t. $\psi(L)$, $L$ are transverse, $\#(\psi(L) \cap L) \geq \sum b_i(L)$.

This is a special case of Arnold’s conjecture: the rough idea is that $H^*(L) \cong HF(L, \psi(L))$ and $\text{rk} CF \geq \text{rk} HF$.

**Example.** Consider $T^*S_1 \cong \mathbb{R} \times S^1$, with $L_0 = \{(0, \theta) \mid \theta \in S^1 = [0, 2\pi]\}$, $L_1 = \{(a \sin \theta + b, \theta)\}$. Then $L_0 \cap L_1 = \{p, q\}$, and the region between them decomposes into disks $u$, $v$. Then $CF(L_0, L_1) = \bigwedge p \oplus \bigwedge q$, $\partial(p) = (\text{area}(u) - \text{area}(v))q$, $\partial(q) = 0$. In this case $(c_1(TM) = 0$, as is the Maslov class of $L_i)$, $\exists a \mathbb{Z}$ grading on $CF$ (because the index is independent of $[u]$), e.g. $\deg p = 0$, $\deg q = 1$. We have two cases:

- if $\text{area}(u) = \text{area}(v)$, then $\partial = 0$, $HF(L_0, L_1) \cong H^*(S^1, \Lambda)$.
- if $\text{area}(u) \neq \text{area}(v)$, then $HF(L_0, L_1) = 0$.

Return to our issues, one can achieve transversality for simple maps by picking $J$ generic, but for multiply covered maps, we need sophisticated techniques such as domain-dependent $J$, multivalued perturbations, virtual cycles, or Kuranishi structures. To obtain an orientation of the moduli space, we need auxiliary data, e.g. a spin structure on $L_0, L_1$. For compactness, the Gromov compactness theorem implies that, given an energy bound, compactness holds after adding limiting configurations. There are three types of phenomena:

- **Bubbling of spheres:** if $|du_n| \to \infty$ at an interior point, the resulting limit is a spherical bubble. The treatment is the same as in Gromov-Witten invariants, and in good cases (if transversality is achieved), the configurations with sphere bubbles have real codimension $\geq 2$ in $\overline{\mathcal{M}}$.
- **Bubbling of disks:** if $|du_n| \to \infty$ at a boundary point, the resulting limit is a disk bubble at the boundary. Even assuming transversality, the space of these will have real codimension 1 in $\overline{\mathcal{M}}$. 

• Breaking of strips: if energy escapes towards \( s \to \pm \infty \), i.e. reparameterizing \( u_n(\cdot - \delta_n, \cdot) \) for \( |\delta_n| \to \infty \) gives different limits, the resulting limit is a sequence of holomorphic strips (that is, what was a single holomorphic strip with progressively thinning “necks” becomes several separate strips).

Finally, we want to have \( \partial^2 = 0 \). Assuming no bubbling, we consider \( \mathcal{M}(p, q, \phi, J)/\mathbb{R} \) for \( J \) generic, \( \phi \in \pi_2, \text{ind}(\phi) = 2 \). We expect a one-dimensional manifold, which is compactified by adding broken trajectories, i.e.

\[
\sqcup r \in L_0 \cap L_1 \quad (\mathcal{M}(p, r, \phi_1, J)/\mathbb{R}) \times (\mathcal{M}(p, r, \phi_2, J)/\mathbb{R})
\]

(3)

\[
\phi_1 \# \phi_2 = \phi
\]

The gluing theorem states that the resulting \( \overline{\mathcal{M}(p, q, \phi, J)}/\mathbb{R} \) is a manifold with boundary. Now, the number of ends of a compact oriented 1-manifold is 0, and thus so are the contributions to the coefficients of \( T^{\omega(\phi)} q \) in \( \partial^2(p) \).