1. Coherent Sheaves on a Complex Manifold (contd.)

We now recall the following definitions from category theory.

**Definition 1.** An additive category is one in which \( \text{Hom}(A,B) \) are abelian groups, composition is distributive, and there is a direct sum \( \oplus \) and a zero object 0. An abelian category is an additive category s.t. every morphism has a kernel and cokernel, e.g. a kernel of \( f : A \to B \) is a morphism \( K \to A \) s.t. \( g : C \to A \) factors through \( K \) uniquely iff \( f \circ g = 0 \).

One can define complexes in an additive category, but one needs to be in an abelian category to have notions of exact sequences and cohomology. Recall that, given chain complexes \( C_*, D_* \), a chain map \( f : C_* \to D_* \) is a collection of maps \( f_i : C_i \to D_i \) commuting with \( \delta \). Given two such maps \( f = \{f_i\}, g = \{g_i\} \), we call them *homotopic* if there is a map \( h : A \to B[-1] \) (\( B \) shifted down by 1) s.t. \( f - b = d_B h + h d_A \), i.e.

\[
\cdots \to A_{i-1} \xrightarrow{d_{i-1}} A_i \xrightarrow{d_i} A_{i+1} \xrightarrow{d_{i+1}} \cdots \\
\xrightarrow{f_{i-1}} \xrightarrow{g_{i-1}} h_{i-1} \xrightarrow{f_i} \xrightarrow{g_i} h_i \xrightarrow{f_{i+1}} \xrightarrow{g_{i+1}} \cdots
\]

A chain map is a *quasi-isomorphism* if the induced maps on cohomology are isomorphisms. This is stronger than \( H^*(C_*) \cong H^*(D_*) \). For \( \mathcal{A} \) an abelian category, the category of bounded chain complexes is the differential graded category whose objects are bounded chain complexes in \( \mathcal{A} \) and whose morphisms are “pre-homomorphisms” of complexes \( \text{Hom}^k(A_*, B_*) = \bigoplus_i \text{Hom}_A(A_i, B_{i+k}) \): it is equipped with a differential \( \delta \) where

\[
f \in \text{Hom}^k(A_*, B_*) \implies \delta(f) = d_B f + (-1)^{k+1} f d_A \in \text{Hom}^{k+1}(A_*, B_*)
\]

Chain maps are precisely the elements of \( \text{Ker}(\delta : \text{Hom}^0 \to \text{Hom}^1) \), and the nullhomotopic maps are elements of \( \text{im}(\delta : \text{Hom}^{-1} \to \text{Hom}^0) \), so \( H^0 \text{Hom}(A, B) \) gives the space of chain maps up to homotopy.

**Definition 2.** For an abelian category, the bounded derived category \( D^b(\mathcal{A}) \) is the triangulated category whose objects are bounded chain complexes in \( \mathcal{A} \) and
whose morphisms are given by chain maps up to homotopy localizing w.r.t. quasi-isomorphisms. That is, quasi-isomorphisms are formally inverted; for any quasi-isomorphism \( s \), we add a morphism \( s^{-1} \). More precisely, \( \text{Hom}_{D^{\text{b}}(A)}(A_*,B_*) = \{ A \leftarrow A' \rightarrow B \}/ \sim \) where \( s \) is a quasi-isomorphism, \( f \) is a chain map, and \( \sim \) is homotopy equivalence. We similarly define the categories \( D^+(A), D^-(A) \) of chain complexes bounded above/below.

To explain the notion of triangulated category, recall the following:

- In the category of topological spaces (or simplicial complexes), there are no kernels and cokernels. Given a map \( f \), however, the mapping cone \( C_f = (X \times [0,1]) \cup Y/(x,0) \sim (x',0), (x,1) \sim f(x) \) acts as both simultaneously. There are natural maps \( i : Y \to C_f \) (inclusion) and \( q : C_f \to \Sigma X \) (collapsing \( Y \)), and we obtain a sequence of topological spaces

\[
X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{q} \Sigma X \to \ldots
\]

with compositions null-homotopic. This gives a long exact sequence of

\[
H_i(X) \to H_i(Y) \to H_i(C_f) \to H_i(\Sigma X) = H_{i-1}(X) \to H_i(\Sigma Y) = H_{i-1}(Y)
\]

- If \( X, Y \) are simplicial complexes, \( f \) a simplicial map, \( C_f \) defined analogously is a simplicial complex, with \( i \)-cells given by cones on \( (i-1) \)-cells of \( X \) and \( i \)-cells of \( Y \). The boundary map is given by the matrix

\[
\left( \begin{array}{cc} \partial X & 0 \\ f & \partial Y \end{array} \right).
\]

- If \( A^* \) and \( B^* \) are complexes, \( f \) a chain map, we define \( C_f = A[1] \oplus B \), i.e. \( C_f^i = A^{i+1} \oplus B^i \). The boundary map is \( \delta = \left( \begin{array}{c} \delta_A[1] \\ f \\ \delta_B \end{array} \right) \). Note that, if \( A, B \) are single objects, Cone\( (f : A \to B) \) is just \{ \( 0 \to A \xrightarrow{i} B \to 0 \) \}. We have natural chain maps \( B^* \xrightarrow{i} C_f^* \) (subcomplex) and \( C_f^* \xrightarrow{q} A^*[1] \) (quotient complex). As before, \( A^*[1] \) is quasi-isomorphic to Cone\( (i : B^* \to C_f^*) \).

- Finally, in the derived category, the inversion of quasi-isomorphisms gives us exact triangles

\[
\begin{array}{ccc}
A^* & \xrightarrow{\cdot} & B^* \\
\downarrow \ & \ & \ & \downarrow \\
C^* & \xrightarrow{\cdot} & A^*[1]
\end{array}
\]

with

\[
H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \ldots
\]
Definition 3. A triangulated category is an additive category with a shift functor [1] and a set of distinguished triangles satisfying various axioms:

- \( \forall X, X \xrightarrow{id} X \to 0 \to X[1] \) is distinguished,
- \( \forall X \to Y, \) there is a distinguished triangle \( X \xrightarrow{u} Y \to Z \to X[1] \) (\( Z \) is called the mapping cone of \( f \)).
- The rotation of any distinguished triangle is distinguished, i.e. \( X \to Y \to Z \to X[1] \) distinguished, \( Y \to Z \to X[1] \to Y[1] \) and \( Z \to X[1] \to Y[1] \to Z[1] \) are distinguished.
- Given a square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{f'} & Y
\end{array}
\]

(7)

there is a map between the mapping cones of \( f, f' \) that makes everything commute in the induced map of distinguished triangles

\[
\begin{array}{ccc}
X & \xrightarrow{} & Y \xrightarrow{} Z \xrightarrow{} X[1] \\
\downarrow & & \downarrow \\
X' & \xrightarrow{} & Y' \xrightarrow{} Z' \xrightarrow{} X'[1]
\end{array}
\]

(8)

- Given a pair of maps \( X \xrightarrow{u} Y \xrightarrow{v} Z \), there are maps between the mapping cones \( C_u, C_v, C_{v\circ u} \) of \( u, v, \) and \( v \circ u \) that make every commute in the induced maps of distinguished triangles.

1.1. Derived functors. Let \( F : \mathcal{A} \to \mathcal{B} \) be a left exact functor between abelian categories. \( \mathcal{R} \subset \mathcal{A} \) is called an adapted class of objects for \( F \) if

- \( \mathcal{R} \) is stable under direct sums,
- for \( C^* \) an acyclic complex of objects in \( \mathcal{R} \), \( F(C^*) \) is acyclic, and
- \( \forall A \in \mathcal{A}, \exists R \in \mathcal{R} \) s.t. \( 0 \to A \xrightarrow{i} R. \)
For instance, the set of injective objects is such an adapted class. Let \( K^+(\mathcal{R}) \) be the homotopy category of complexes bounded below of objects in \( \mathcal{R} \). \( RF \) gives a composition \( D^+(A) \to K^+(\mathcal{R}) \xrightarrow{F} D^+(B) \), where the first map is induced by resolution by objects of \( R \). The map \( D^+(A) \to D^+(B) \) is exact, i.e. it maps exact triangles to exact triangles, and \( R^iF = H^i(RF). \)

1.2. Extensions. Let \( A, B \in \mathcal{A} \hookrightarrow D^b(\mathcal{A}) \) be single object complexes concentrated in degree 0, so \( B[k] \) is concentrated in degree \( -k \).

**Proposition 1.** \( \text{Hom}_{D^b(\mathcal{A})}(A, B[k]) = \text{Ext}_A^k(A, B). \)

We can use this to define a product \( \text{Ext}_A^k(A, B) \otimes \text{Ext}_A^\ell(B, C) \to \text{Ext}_A^{k+\ell}(A, C) \) as a composition \( A \to B[k] \to C[k+\ell] \) in \( D^b(\mathcal{A}) \).

**Example.** For \( k = 1 \), we have

\[
\begin{array}{c}
0 & \to & 0 & \to & A & \to & 0 \\
\downarrow & & & & \downarrow & & \\
0 & \to & B & \to & 0 & \to & 0
\end{array}
\]

(10)

There are no chain maps, but we can invert quasi-isomorphisms. If we have an extension \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) in \( \mathcal{A} \), we have chain maps

\[
\begin{array}{c}
0 & \to & 0 & \to & C & \to & 0 \\
\downarrow & & & & \uparrow & & \\
0 & \to & A & \xrightarrow{id} & B & \to & 0
\end{array}
\]

(11)

giving an element in \( \text{Hom}_{D^b(\mathcal{A})}(C, A[1]) = \text{Ext}_A^1(C, A) \).

There are two ways to understand the above proposition. First, if \( \mathcal{A} \) has enough injectives, take a resolution of \( B \) by a complex \( I^0 \to I^1 \to \cdots \) quasi-isomorphic to \( B \): the chain maps from \( A \) to \( I^* \) are, up to homotopy, isomorphic to \( H^k(\text{Hom}(A, I^*)) \cong \text{Ext}_A^k(A, B). \) Second, we can check the definition of a derived functor. Given a short exact sequence \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) in \( \mathcal{A} \), we get an exact triangle \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{w} A[1] \) quasi-isomorphic to a distinguished triangle with \( \text{Cone}(f) \).
**Proposition 2.** For an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ and an object $E$, we have long exact sequences

\[\cdots \rightarrow \text{Hom}(E, A[i]) \xrightarrow{f} \text{Hom}(E, B[i]) \xrightarrow{g} \text{Hom}(E, C[i]) \xrightarrow{h} \text{Hom}(E, A[i + 1]) \rightarrow \cdots\]

\[\cdots \rightarrow \text{Hom}(A[i + 1], E) \xrightarrow{h^*} \text{Hom}(C[i], E) \xrightarrow{g^*} \text{Hom}(B[i], E) \xrightarrow{f^*} \text{Hom}(A[i], E) \rightarrow \cdots\]