1. Derived Fukaya Category

Last time: derived categories for abelian categories (e.g. $D^b\text{Coh}(X)$). This time: the derived Fukaya category. We start with an $A_\infty$-category $\mathcal{A}$ and obtain a triangulated category via “twisted complexes”. Recall that in an $A_\infty$-category, $\text{hom}_A(X,Y)$ is a graded vector space equipped with maps

1) 

$$m_k : \text{hom}_A(X_0, X_1) \otimes \cdots \otimes \text{hom}_A(X_{k-1}, X_k) \to \text{hom}_A(X_0, X_k)[2-k]$$

Additive enlargement: we define the category $\Sigma \mathcal{A}$ to be the category whose objects are finite sums $\bigoplus_i X_i[k_i], X_i \in \mathcal{A}, k_i \in \mathbb{Z}$ and whose maps are

2) 

$$\text{hom}_{\Sigma \mathcal{A}}(\bigoplus_i X_i[k_i], \bigoplus_j Y_j[\ell_j]) = \bigoplus_{i,j} \text{hom}_\mathcal{A}(X_i, Y_j)[\ell_j - k_i]$$

Note that we have induced multiplication maps

3) 

$$m_k(a_k, \ldots, a_1)^{i,j} = \sum_{i_1, \ldots, i_{k-1}} m_k(a_k^{i_k-1,j}, \ldots, a_1^{i_1,j})$$

2) Twisted complexes: we define the category $\text{Tw} \mathcal{A}$ to be the category whose objects are twisted complexes $(X, \delta_X)$,

4) 

$$X = \bigoplus_i X_i[k_i] \in \Sigma \mathcal{A}, \delta_X = (\delta_X^{ij}) \in \text{hom}_{\Sigma \mathcal{A}}^1(X, X)$$

(i.e. $\delta_X$ a degree 1 endomorphism) s.t.

- $\delta_X$ is strictly lower-triangular, and
- $\sum_{k=1}^\infty m_k(\delta_x, \ldots, \delta_x) = 0$. It is a finite sum because $\delta_X$ is lower triangular, and generalizes $\delta_X \circ \delta_X = 0$.

Example. For a simple map $f : X_1 \to X_2, f \in \text{hom}_A^1(X_1, X_2)$, the condition is $m_1(f) = 0$. Now, for maps $X_1[2] \xrightarrow{f} X_2[1] \xrightarrow{g} X_3$ and $X_1[2] \xrightarrow{h} X_3$,

- $g \in \text{hom}^0(X_2, X_3) = \text{hom}^1(X_2[1], X_3)$
- $f \in \text{hom}^0(X_1[1], X_2[1]) = \text{hom}^1(X_1[2], X_2[1])$
- $h \in \text{hom}^{-1}(X_1, X_3) = \text{hom}^1(X_1[2], X_3)$

the condition is $m_1(f) = m_1(g) = 0$ and $m_2(g, f) + m_1(h) = 0$. 

The morphisms in the category of twisted complexes are

\[ \text{hom}_{\text{Tw}, \mathcal{A}}((X, \delta_X), (Y, \delta_Y)) = \text{hom}_{\Sigma \mathcal{A}}(X, Y) \]

and

\[
m^k_{\text{Tw}, \mathcal{A}}(a_k, \ldots, a_1) = \sum_{i_0, \ldots, i_k} \pm m^k_{\Sigma \mathcal{A}} \left( \delta_{X_{i_k}}, \ldots, \delta_{X_{i_0}}, a_k, \delta_{X_{i_{k-1}}}, \ldots, \delta_{X_{i_1}}, \ldots, \delta_{X_{i_1}}, a_1, \delta_{X_{i_0}}, \ldots, \delta_{X_0} \right)
\]

Tw\mathcal{A} is a triangulated \(A_\infty\)-category, i.e. there are mapping cones satisfying the usual axioms.

**Example.** For \(a \in \text{hom}(X, Y)\),

\[ m^1_{\text{Tw}, \mathcal{A}}(a) = m_1(a) \pm m_2(\delta_Y, a) \pm m_2(a, \delta_X) + \text{higher terms} \]

This is a generalization of being a chain map up to homotopy.

3) We now take the cohomology category \(D(\mathcal{A}) := H^0(\text{Tw}, \mathcal{A})\), which is an honest triangulated category. The objects of the two categories are the same, but now our morphisms are \(\text{hom}^{D(\mathcal{A})}(X, Y) := H^0(\text{hom}^{\text{Tw}, \mathcal{A}}(X, Y), m^1_{\text{Tw}, \mathcal{A}})\). Note that \(\text{hom}^{D(\mathcal{A})}(X, Y[k]) = H^k(\text{hom}^{\text{Tw}, \mathcal{A}}(X, Y), m^1_{\text{Tw}, \mathcal{A}})\). The composition is induced by \(m^2_{\text{Tw}, \mathcal{A}}\) on cohomology.

**Remark.** There is a variant of this called a split-closed derived category. Let \(\mathcal{A}\) be a linear category, \(X \in \mathcal{A}, p \in \text{hom}_\mathcal{A}(X, X)\) s.t. \(p^2 = p\) (idempotent). Define the image of \(p\) to be an object \(Y\), and add maps \(u : X \to Y, v : Y \to X\) s.t. \(u \circ v = \text{id}_Y, v \circ u = p\). That is, we enlarge \(\mathcal{A}\) to add these objects and maps, and define the split closure to be the category whose objects are \((X, p)\) with \(p\) idempotent, and morphisms \(\text{hom}((X, p), (Y, p')) = p' \text{hom}(X, Y)p\). This is more complicated in the \(A_\infty\) setting.

Geometrically, some exact triangles in \(DFuk(M)\) are given by Lagrangian connected sums (FOOO) and Dehn twists (Seidel).

- For an example of the latter, given a cylinder with a Lagrangian circle \(S\), we can obtain a symplectomorphism \(\tau_S \in \text{Symp}(M, \omega)\) which is the identity outside a neighborhood of \(S\) and, within that neighborhood, twists the cylinder around (in higher dimensions, define this using the geodesic flow in a neighborhood of \(S \cong T^*S\)). If \(L\) is Lagrangian, then \(\tau_S(L)\) is Lagrangian as well. By Seidel, there exists an exact triangle in
$DFuk(M)$:

\[
\begin{array}{c}
HF^*(S, L) \otimes S \xrightarrow{t} L \\
\downarrow \tau_S(L) \\
[1]
\end{array}
\]

(9)

These correspond to long exact sequences for $HF(L', -)$.

- In the former situation, for $L_1, L_2$ (graded) Lagrangians, $L_1 \cap L_2 = \{p\}$ of index 0, we can construct the connected sum $L_1 \#_p L_2$, which looks locally like $\tau_{L_1}(L_2)$ if $L_1$ is a sphere and is given by $\text{Cone}(L_1 \overset{p}{\to} L_2)$ in general (consider this vs. “$L_1 \cup_p L_2 \simeq \text{Cone}(L_1 \overset{0}{\to} L_2)$”). For instance, in the torus $T^2$, consider two independent loops $\alpha$ of degree 2 and $\beta$ of degree 1, with two points of intersection $p, q$. Then $\text{Cone}(\alpha \overset{p+q}{\to} \beta) \simeq \gamma_1 \oplus \gamma_2$ is disconnected, where $\gamma_1, \gamma_2$ are degree 1 loops. If we only started with $\alpha, \beta$, the triangulated envelope contains $\gamma_1 \oplus \gamma_2$, but not $\gamma_1, \gamma_2$ separately. The split-closure does contain them.

- Now, if we start with two independent generators of the torus, successive Dehn twists give all the homotopy classes of loops in $T^2$, but each homotopy class contains infinitely many non-Hamiltonian isotopic Lagrangians. To generate $DFuk(T^2)$ as a triangulated envelope, we need (for instance) one horizontal loop and infinitely many vertical loops. On the other hand, $\alpha, \beta$ above are split generators. The key point is that $\text{Cone}(\alpha \overset{p+T^q}{\to} \beta)$ gives deformed loops, direct sums of which vary continuously within a homotopy class. But many cones and idempotents have no obvious geometric interpretation. For instance, the Clifford torus $T = \{ |x| = |y| = |z| \} \subset \mathbb{CP}^2$ has idempotents in $HF(T, T)$ without any obvious geometric interpretation.