1. SYZ Conjecture (cntd.)

Recall:

**Proposition 1.** First order deformations of special Lagrangian $L$ in a strict (resp. almost) Calabi-Yau manifold are given by $\mathcal{H}^1(L, \mathbb{R})$ (resp. $\mathcal{H}^1_\psi(L, \mathbb{R})$), where

$$
H^1_\psi(L, \mathbb{R}) = \{ \beta \in \Omega^1(L, \mathbb{R}) \mid d\beta = 0, d^*(\psi \beta) = 0 \}
$$

It is still true that $\mathcal{H}^1_\psi(L, \mathbb{R}) \cong H^1(L, \mathbb{R})$.

**Theorem 1** (McLean, Joyce). Deformations of special Lagrangians are unobstructed, i.e. the moduli space of special Lagrangians is a smooth manifold $B$ with $T_L B \cong \mathcal{H}^1_\psi(L, \mathbb{R}) \cong H^1(L, \mathbb{R})$.

There are two canonical isomorphisms $T_L B \cong H^1(L, \mathbb{R}), v \mapsto [-v \omega]$ ("symplectic") and $T_L B \cong H^{n-1}(L, \mathbb{R}), v \mapsto [v, \text{Im } \Omega]$ "complex".

**Definition 1.** An affine structure on a manifold $N$ is a set of coordinate charts with transition functions in $GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$.

**Corollary 1.** $B$ carries two affine structures.

For affine manifolds, mirror symmetry exchanges the two affine structures. Our particular case of interest is that of special Lagrangian tori, so dim $H^1 = n$. The usual harmonic 1-forms on flat $T^n$ have no zeroes, and give a pointwise basis of $T^* L$. We will make a standing assumptions that $\psi$-harmonic 1-forms for $g|_L$ have no zeroes (at least ok for $n \leq 2$). Then a neighborhood of $L$ is fibered by special Lagrangian deformations of $L$: locally,

$$
\begin{align*}
T^n & \longrightarrow U \subset X \\
& \downarrow \pi \\
V & \subset B
\end{align*}
$$

(2)

In local affine coordinates, we pick a basis $\gamma_1, \ldots, \gamma_n \in H_1(L, \mathbb{Z})$: deforming from $L$ to $L'$, the deformation of $\gamma_i$ gives a cylinder $\Gamma_i$, and we set $x_i = \int_{\Gamma_i} \omega$ (the flux of the deformation $L \rightarrow L'$). These are affine coordinates on the symplectic
side. On the complex side, pick a basis $\gamma_1^*, \ldots, \gamma_n^* \in H_{n-1}(L, \mathbb{Z})$, construct the associated $\Gamma_i^*$, and set $x_i^* = \int_{\Gamma_i} \Im \Omega$. Globally, there is a monodromy $\pi_1(B, \ast) \to \text{Aut} H^*(L, \mathbb{Z})$. In our case, the monodromies in $GL(H^1(L, \mathbb{Z})), GL(H^{n-1}(L, \mathbb{Z}))$ are transposes of each other.

1.1. Prototype construction of a mirror pair. Let $B$ be an affine manifold, $\Lambda \subset TB$ the lattice of integer vectors. Then $TB/\Lambda$ is a torus bundle over $B$, and carries a natural complex structure, e.g.

$$T(\mathbb{R}^n) \cong \mathbb{C}^n, \mathbb{C}^n = \mathbb{R}^n \oplus \mathbb{R}^n, GL(n, \mathbb{Z}) \ni A \mapsto \begin{pmatrix} A & 0 \\ 0 & A^T \end{pmatrix} \in \text{Sp}(2n)$$

Setting $\Lambda^* = \{ p \in T^*B \mid p(\Lambda) \subset \mathbb{Z} \}$ to be the dual lattice of integer covectors, we find that $T^*B/\Lambda^*$ has a natural symplectic structure since $GL(n, \mathbb{Z}) \ni A \mapsto \begin{pmatrix} A & 0 \\ 0 & A^T \end{pmatrix} \in \text{Sp}(2n)$.

In our case, we have two affine structures with dual monodromies

$$
\begin{array}{ccc}
TB & \sim & T^*B \\
\sim & \downarrow \text{cx} & \sim \\
H^{n-1}(L, \mathbb{R}) & \sim & H_1(L, \mathbb{R}) \\
\downarrow \text{PD} & & \downarrow \\
\Lambda_c & = & H^{n-1}(L, \mathbb{Z}) \sim H_1(L, \mathbb{Z}) = \Lambda^*_s
\end{array}
$$

so the complex manifold $TB/\Lambda_c$ is diffeomorphic to the symplectic manifold $T^*B/\Lambda^*_s$. Dually, $X^\vee \cong T^*B/\Lambda^*_c \cong TB/\Lambda_s$.

1.2. More explicit constructions [cf. Hitchin]. Let

$$M = \{(L, \nabla) \mid L \text{ a special Lagrangian torus in } X, \nabla \text{ flat } U(1) - \text{conn on } \mathbb{C} \times L \text{ mod gauge}\}$$

i.e. $\nabla = d + iA, iA \in \Omega^1(L, i\mathbb{R}), dA = 0$ mod exact forms.

$$T_{(L, \nabla)}M = \{(v, i\alpha) \in C^\infty(NL) \oplus \Omega^1(L, i\mathbb{R}) \mid - \iota_v \omega \in \mathcal{H}^1_\psi(L, \mathbb{R}), d\alpha = 0 \text{ mod Im } (d)\}$$

$$= \{(v, i\alpha) \in C^\infty(NL) \oplus \Omega^1(L, i\mathbb{R}) \mid - \iota_v \omega + i\alpha \in \mathcal{H}^1_\psi(L; \mathbb{C})\}$$

$$= H^1_\psi(L, \mathbb{C})$$

which is a complex vector space, and $J^\vee$ is an almost-complex structure.

**Proposition 2.** $J^\vee$ is integrable.
Proof. We build local holomorphic coordinates. Let \( \gamma_1, \ldots, \gamma_n \) be a basis of \( H_1(L, \mathbb{Z}) \), and assume \( \gamma_i = \partial \beta_i, \beta_i \in H_2(X, L) \). Set

\[
(7) \quad z_i(L, \nabla) = \exp(-\int_{\beta_i} \omega \mid_{\mathbb{C}^*}) \in \mathbb{C}^*
\]

Then

\[
(8) \quad \text{dlog} \, z_i : (v, i\alpha) \mapsto -\int_{\gamma_i} \iota_v \omega + \int_{\gamma_i} i\alpha = \langle [-\iota_v \omega + i\alpha], \gamma_i \rangle_{H^1(L, \mathbb{C})}
\]

is \( \mathbb{C} \)-linear. If there are no such \( \beta_i \), we instead use a deformation tube as constructed earlier. Warning: all of our formulas are up to (i.e. may be missing) a factor of \( 2\pi \).

Next, consider the holomorphic \((n, 0)\)-form on \( M \)

\[
(9) \quad \Omega^\vee((v_1, i\alpha_1), \ldots, (v_n, i\alpha_n)) = \int_L (-\iota_{v_1} \omega + i\alpha_1) \wedge \cdots \wedge (-\iota_{v_n} \omega + i\alpha_n)
\]

After normalizing \( \int_L \Omega = 1 \), we have a Kähler form

\[
(10) \quad \omega^\vee((v_1, i\alpha_1), (v_2, i\alpha_2)) = \int_L \alpha_2 \wedge (\iota_{v_1} \text{Im} \, \Omega) - \alpha_1 \wedge (\iota_{v_2} \text{Im} \, \Omega)
\]

**Proposition 3.** \( \omega^\vee \) is a Kähler form compatible with \( J^\vee \).

*Proof.* Pick a basis \( [\gamma_i] \) of \( H_{n-1}(L, \mathbb{Z}) \) with a dual basis \( [e_i] \) of \( H_1(L, \mathbb{Z}) \), i.e. \( e_i \cap \gamma_j = \delta_{ij} \). For all \( a \in H^1(L), b \in H^{n-1}(L) \)

\[
(11) \quad \langle a \cup b, [L] \rangle = \sum_i \langle a, e_i \rangle \langle b, \gamma_i \rangle
\]

Letting \( a = \sum a_i dx_i, b = \sum b_i (-1)^{i-1} (dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n), \int_{\gamma_i} a \wedge b = \sum a_i b_i \). Again, take a deformation from \( L_0 \) to \( L' \), \( C_i \) the tube (an \( n \)-chain) formed by the deformation of \( \gamma_i \), and set \( p_i = \int_{C_i} \text{Im} \, \Omega, \theta_i = \int_{e_i} A \) for \( A \) the connection 1-form (i.e. \( \text{hol}_{e_i}(\nabla) = \exp(i\theta_i) \)). Then

\[
(12) \quad dp_i : (v, i\alpha) \mapsto \int_{\gamma_i} \iota_v \text{Im} \, \Omega = \langle [\iota_v \text{Im} \, \Omega], \gamma_i \rangle
\]

\[
\quad d\theta_i : (v, i\alpha) \mapsto \int_{e_i} \alpha = \langle [\alpha], e_i \rangle
\]
By (11), $\omega^\vee = \sum dp_i \wedge d\theta_i$, implying that $\omega^\vee$ is closed, and

$$\omega^\vee((v_1, \alpha_1), (v_2, \alpha_2)) = \int_L \alpha_2 \wedge (-\psi \ast_g \iota_{v_1} \omega) - \alpha_1 \wedge (-\psi \ast_g \iota_{v_2} \omega)$$

(13)

$$\omega^\vee((v_1, \alpha_1), J^\vee(v_2, \alpha_2)) = \int_L \psi \cdot ((\alpha_1, \iota_{v_2} \omega)_g - (\alpha_2, \iota_{v_1} \omega)_g) \text{vol}_g$$

$$\omega^\vee((v_1, \alpha_1), J^\vee(v_2, \alpha_2)) = \int_L \psi \cdot ((\alpha_1, \iota_{v_2} \omega)_g + (\iota_{v_2} \omega, \iota_{v_2} \omega)_g) \text{vol}_g$$

which is clearly a Riemannian metric. \qed