Last time, we were considering \( \mathbb{CP}^1 \) mirror to \( \mathbb{C}^* \), \( W = z + \frac{e^{-\lambda}}{z} \) for \( \lambda = 2\pi \int_{\mathbb{CP}^1} \omega \): the latter object is a Landau-Ginzburg model, i.e. a Kähler manifold with a holomorphic function called the “superpotential”. Homological mirror symmetry gave

\[
D^\bullet \text{Fuk}(\mathbb{CP}^1) \cong H^0 \text{MF}(W)
\]

\[
D^\bullet \text{Coh}(\mathbb{CP}^1) \cong D^0 \text{Fuk}(\mathbb{C}^*, W)
\]

We stated that the Fukaya category of \( \mathbb{CP}^1 \) was a collection indexed by “charge” \( \lambda \in \mathbb{C} \), and defined \( \text{Fuk}(\mathbb{CP}^1, \lambda) \) to be the set of weakly unobstructed Lagrangians with \( m_0 = \lambda \cdot [L] \). This is an honest \( A_{\infty} \)-category, as the \( m_0 \)'s cancel and the Floer differential squares to zero, whereas from \( \lambda \) to \( \lambda' \) we’d have \( \partial^2 = \lambda' - \lambda \). For instance, for \( L \cong S^1 \), \( (L, \nabla) \) is weakly unobstructed, with \( m_0 = W(L, \nabla) \cdot [L] \). However, \( HF(L, L) = 0 \) unless \( L \) is the equator and \( \text{hol}(\nabla) = \pm \text{id} \). Then \( L_\pm \) has \( HF \cong H^*(S^1, \mathbb{C}) \) with deformed multiplicative structure, \( HF^*(L, L) \cong \mathbb{C}[t]/t^2 = \pm e^{-\lambda/2} \).

We now look at the matrix factorizations of \( W - \lambda, \lambda \in \mathbb{C} \). These are \( \mathbb{Z}/2\mathbb{Z} \)-graded projective modules \( Q \) over the ring of Laurent polynomials \( R = \mathbb{C}[\mathbb{C}^*] \cong \mathbb{C}[z^{\pm 1}] \) equipped with \( \delta \in \text{End}^1(Q) \) s.t. \( \delta^2 = (W - \lambda) \cdot \text{id}_Q \). That is, we have maps \( \delta_0 : Q_0 \to Q_1, \delta_1 : Q_1 \to Q_0 \) given by matrices with entries in the space of Laurent polynomials s.t. \( \delta_0 \circ \delta_1 = (W - \lambda) \cdot \text{id}_{Q_1}, \delta_1 \circ \delta_0 = (W - \lambda) \cdot \text{id}_{Q_0} \). Now \( \text{Hom}(Q, Q') \) is \( \mathbb{Z}/2\mathbb{Z} \) graded, with

\[
\text{Hom}^0 = \{ \begin{array}{c}
Q_0 \\
Q_0'
\end{array} \xrightarrow{\begin{array}{c}
\delta_0 \\
\delta_1
\end{array}} \begin{array}{c}
Q_1 \\
Q_1'
\end{array} \}
\]

\[
f_0 \\
f_1
\]

This has a differential \( \partial \) s.t. \( \partial(f) = \delta' \cdot f \pm f \cdot \delta \) and \( \partial^2 = 0 \). We obtain a homology category \( H^0 \text{MF}(W - \lambda) \): \( \text{hom} = H^0(\text{Hom}, \partial) \), i.e. “chain maps” up to “homotopy”.

**Theorem 1.** \( H^0(\text{MF}(W - \lambda)) = 0 \), i.e. all matrix factorizations are nullhomotopic, unless \( \lambda \) is a critical value of \( W \).
Warning: again, looking at homomorphisms from $MF(W - \lambda)$ to $MF(W - \lambda')$, then $\partial^2 \neq 0$, $\partial^2(f) = \partial^2 f - f \cdot \partial^2 = (\lambda - \lambda')f$.

Example. $W = z + \frac{e^{-\lambda}}{z}$ has critical points $\pm e^{-\Lambda/2}$ with critical values $\pm 2e^{-\Lambda/2}$. Then

$$W \pm 2e^{-\Lambda/2} = z \pm 2e^{-\Lambda/2} + \frac{e^{-\lambda}}{z} = (z \pm e^{-\Lambda/2})(1 \pm \frac{e^{-\Lambda/2}}{z})$$

(3)

$$Q_\pm = \{ \mathbb{C}[z^{\pm 1}] \xrightarrow{\frac{z \pm e^{-\Lambda/2}}{1 \pm e^{-\Lambda/2}z^{-1}}} \mathbb{C}[z^{\pm 1}] \}$$

Then

$$\text{End}_{H^0MF}(Q_\pm) = \{ R \xrightarrow{f} R \} / \text{homotopy}$$

(4)

is multiplication by $f \in \mathbb{C}[z^{\pm 1}]$. The maps $\partial$ sends

$$R \xrightarrow{h} R \mapsto R \xrightarrow{(x \pm e^{-\Lambda/2})h} R \xrightarrow{f} R \xrightarrow{f} R$$

(5)

and similarly on the other side, so

$$\text{End}(Q_\pm) = \mathbb{C}[z^{\pm 1}]/(z \pm e^{-\Lambda/2}, 1 \pm e^{-\Lambda/2}z^{-1}) \cong (\mathbb{C}[z^{\pm 1}]/z \pm e^{-\Lambda/2}) \cong \mathbb{C}$$

(6) $\text{Hom}_{H^0MF}(Q_\pm, Q_{\pm[1]}) \cong \mathbb{C}$.

Similarly $\text{Hom}_{H^0MF}(Q_\pm, Q_{\pm[1]}) \cong \mathbb{C}$.

Indeed, in the case of the two maps $z - c, 1 - cz^{-1}$, we take vertical maps $z, 1$, so

$$R \xrightarrow{z - c} R \xrightarrow{z} R \xrightarrow{1 - cz^{-1}} R \xrightarrow{1 - cz^{-1}} R$$

(7)

giving us $\mathbb{C}[z^{\pm 1}]/(z - c)$.

Next, $D^b\text{Coh}(\mathbb{CP}^1)$ is generated by $O(-1)$ and $O$, i.e. the smallest full subcategory containing $O, O(-1)$ and closed under shifts and cones contains all of $D^b$. More generally, via Beilinson we have that

$$D^b\text{Coh}(\mathbb{CP}^n) = \langle O(-n), \ldots, O(-1), O \rangle$$

(8)

The idea is the diagonal $\Delta \subset \mathbb{CP}^n \times \mathbb{CP}^n$ is the (transverse) zero set of $s = \sum_{i=0}^n \frac{\partial}{\partial x_i} \otimes y_i$, which is a section of $E = T(-1) \otimes O(1) = \pi_1(T\mathbb{CP}^n \otimes \mathbb{CP}^n)$.
\(O(-1) \otimes \pi_2^* O(1)\). Recall that \(T\mathbb{CP}^n\) is spanned by the vector fields \(x_i \frac{\partial}{\partial x_i}\) on \(\mathbb{C}^{n+1}\) under the relation \(\sum_{i=0}^n x_i \frac{\partial}{\partial x_i} = 0\). Taking the Koszul resolution

\[
0 \to E^* = \Omega^1(1) \otimes O(-1) \to O \otimes O \to O_\Delta \to 0
\]

in \(D^b \text{Coh}(\mathbb{P}^1 \times \mathbb{P}^1)\). On the other hand, \(E \in D^b \text{Coh}(X \times Y)\) gives \(\phi^E : D^b (\text{Coh}(X) \to D^b \text{Coh}(Y), \mathcal{F} \mapsto R\pi_2^*(L\pi^*_1 \mathcal{F} \otimes L \otimes \mathcal{E})\). Exactness implies that \(\phi^{O_\Delta}(\mathcal{F}) \cong \mathcal{F}\) sits in an exact triangle with

\[
\phi^{O \otimes O(-1)}(\mathcal{F}) \cong R\Gamma(\mathcal{F} \otimes \Omega^1(1)) \otimes_C O(-1)
\]

(10)

\[
\phi^{O \otimes O}(\mathcal{F}) \cong R\Gamma(\mathcal{F}) \otimes_C O
\]

which completes the proof.

The algebra of the exceptional collection \((O(-1), O)\) is given by

\[
\mathcal{A} = \text{End}^*(O(-1) \oplus O)
\]

and \(D^B \text{Coh}(\mathbb{CP}^1)\) is isomorphic to the derived category of finitely-generated \(\mathcal{A}\)-modules.

Finally, the Fukaya category of \((\mathbb{C}^*, W = z + \frac{e^{\Lambda}}{2})\) is the category whose objects are admissible Lagrangians with flat connections, i.e. \(L\) is a (possibly noncompact) Lagrangian submanifold with \(W|_L\) proper, \(W|_L \in \mathbb{R}_+\) outside a compact subset. We can perturb such \(L\): for \(a \in \mathbb{R}\), let \(L^{(a)}\) be Hamiltonian isotopic to \(L, W(L^{(a)}) \in \mathbb{R}_+ + ia\) near \(\infty\). In good cases, it will be the Hamiltonian flow of \(X_{\text{Re} W} = \nabla \text{Im} W\). Then \(\text{Hom}(L, L') = CF^*(L^{(a)}, L^{(a')})\) for \(a > a'\) (the Floer differential is well-defined), and we obtain \(m_k, k \geq 2\) similarly, perturbing the Lagrangians so they are in decreasing order of \(\text{Im} (W)\).

**Example.** Consider \(L_0 = \mathbb{R}_+, L_{-1} = \) an arc joining 0 to \(+\infty\) and rotating once clockwise around the origin. Then \(e^{-\Lambda/2} \in L_0, -e^{-\Lambda/2} \in L_{-1}\), so under \(W = z + \frac{e^{\Lambda}}{2}\), we have \(W(L_0)\) being the interval \([2e^{-\Lambda/2}, +\infty)\) on the positive real axis, while \(W(L_{-1})\) is an arc that joins \(-2e^{-\Lambda/2}\) to \(+\infty\) in the lower half plane. Furthermore, \(\text{hom}(L_0, L_{-1}) \cong \mathbb{C} \cdot e, e = \text{id}_{L_0}\), and same for \(L_{-1}\), while \(\text{hom}(L_0, L_{-1}) = 0\) and \(\text{hom}(L_{-1}, L_0) = V\) has dimension 2. Then \(\text{Fuk}(\mathbb{C}^*, W)\) is generated by \(L_{-1}, L_0\) (Seidel).

Similarly, one can obtain homological mirror symmetry for toric Fano manifolds: see M. Abouzaid.