Chapter 18

Weierstrass-Enneper Representations

18.1 Weierstrass-Enneper Representations of Minimal Surfaces

Let $M$ be a minimal surface defined by an isothermal parameterization $x(u, v)$. Let $z = u + iv$ be the corresponding complex coordinate, and recall that
\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)
\]
Since $u = 1/2(z + \bar{z})$ and $v = -i/2(z - \bar{z})$ we may write
\[
x(z, \bar{z}) = (x^1(z, \bar{z}), x^2(z, \bar{z}), x^3(z, \bar{z}))
\]

Let $\phi = \frac{\partial x}{\partial z}, \phi^i = \frac{\partial x^i}{\partial z}$. Since $M$ is minimal we know that $\phi^i$s are complex analytic functions. Since $x$ is isothermal we have
\[
(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 0 \quad \text{(18.1)}
\]
\[
(\phi^1 + i\phi^2)(\phi^1 - i\phi^2) = -((\phi^3)^2) \quad \text{(18.2)}
\]
Now if we let \( f = \phi^1 - i\phi^2 \) and \( g = \phi^3/(\phi^1 - i\phi^2) \) we have

\[
\begin{align*}
\phi^1 &= 1/2f(1 - g^2), \\
\phi^2 &= i/2f(1 + g^2), \\
\phi^3 &= fg
\end{align*}
\]

Note that \( f \) is analytic and \( g \) is meromorphic. Furthermore \( fg^2 \) is analytic since \( fg^2 = -(\phi^1 + i\phi^2) \). It is easy to verify that any \( \phi \) satisfying the above equations and the conditions of the preceding sentence determines a minimal surface. (Note that the only condition that needs to be checked is isothermality.) Therefore we obtain:

**Weierstrass-Enneper Representation I** If \( f \) is analytic on a domain \( D \), \( g \) is meromorphic on \( D \) and \( fg^2 \) is analytic on \( D \), then a minimal surface is defined by the parameterization \( x(z, \overline{z}) = (x^1(z, \overline{z}), x^2(z, \overline{z}), x^3(z, \overline{z})) \), where

\[
\begin{align*}
x^1(z, \overline{z}) &= \text{Re} \int f(1 - g^2)dz & (18.3) \\
x^2(z, \overline{z}) &= \text{Re} \int if(1 + g^2)dz & (18.4) \\
x^3(z, \overline{z}) &= \text{Re} \int fgdz & (18.5)
\end{align*}
\]

Suppose in WERI \( g \) is analytic and has an inverse function \( g^{-1} \). Then we consider \( g \) as a new complex variable \( \tau = g \) with \( d\tau = g'dz \) Define \( F(\tau) = f/g' \) and obtain \( F(\tau)d\tau = fdz \). Therefore, if we replace \( g \) with \( \tau \) and \( fdz \) with \( F(\tau)d\tau \) we get

**Weierstrass-Enneper Representation II** For any analytic function \( F(\tau) \), a minimal surface is defined by the parameterization \( x(z, \overline{z}) = (x^1(z, \overline{z}), x^2(z, \overline{z}), x^3(z, \overline{z})) \),
where

\[
x^1(z, \bar{z}) = Re \int F(\tau)(1 - \tau^2)dz \quad (18.6)
\]

\[
x^2(z, \bar{z}) = Re \int iF(\tau)(1 + \tau^2)dz \quad (18.7)
\]

\[
x^3(z, \bar{z}) = Re \int F(\tau)\tau dz \quad (18.8)
\]

This representation tells us that any analytic function \( F(\tau) \) defines a minimal surface.

**class exercise** Find the WERI of the helicoid given in isothermal coordinates \((u, v)\)

\[
x(u, v) = (\sinh \sin v, -\sinh \cos v, -v)
\]

Find the associated WERII. (answer: \(i/2\tau^2\)) Show that \( F(\tau) = 1/2\tau^2 \) gives rise to catenoid. Show moreover that \( \tilde{\phi} = -i\phi \) for conjugate minimal surfaces \( x \) and \( \tilde{x} \).

**Notational convention** We have two \( F \)s here: The \( F \) of the first fundamental form and the \( F \) in WERII. In order to avoid confusion well denote the latter by \( T \) and hope that Oprea will not introduce a parameter using the same symbol. Now given a surface \( x(u, v) \) in \( R^3 \) with \( F = 0 \) we make the following observations:

i. \( x_u, x_v \) and \( N(u, v) \) constitute an orthogonal basis of \( R^3 \).

ii. \( N_u \) and \( N_v \) can be written in this basis coefficients being the coefficients of matrix \( dNp \)

iii. \( x_u u, x_v v \) and \( x_u v \) can be written in this basis. One should just compute the dot products \( \langle x_u, x_u \rangle, \langle x_u, x_v \rangle, \langle x_u, N \rangle \) in order to represent \( x_{uu} \) in this basis. The same holds for \( x_{uv} \) and \( x_{vv} \). Using the above ideas one gets the
following equations:

\[
x_{uu} = \frac{E_u}{2E} x_u - \frac{E_v}{2G} + eN \tag{18.9}
\]

\[
x_{uv} = \frac{E_v}{2E} x_u + \frac{G_v}{2G} + fN \tag{18.10}
\]

\[
x_{vv} = -\frac{G_u}{2E} x_u + \frac{G_v}{2G} + gN \tag{18.11}
\]

\[
N_u = -\frac{e}{E} x_u - \frac{f}{G} x_v \tag{18.12}
\]

\[
N_v = -\frac{f}{E} x_u - \frac{g}{G} x_v \tag{18.13}
\]

Now we state the Gauss theorem egregium:

**Gauss Theorem Egregium** The Gauss curvature \( K \) depends only on the metric \( E, F = 0 \) and \( G \):

\[
K = -\frac{1}{2\sqrt{EG}} \left( \frac{\partial}{\partial v} \left( \frac{E_v}{\sqrt{EG}} \right) + \frac{\partial}{\partial u} \left( \frac{G_u}{\sqrt{EG}} \right) \right)
\]

This is an important theorem showing that the isometries do not change the Gaussian curvature.

**proof** If one works out the coefficient of \( x_v \) in the representation of \( x_{uu} - x_{uvu} \) one gets:

\[
x_{uu} = \left[ x_u + \frac{E_u G_u}{4EG} - \left( \frac{E_v}{2G} \right)_u - \frac{E_v G_v}{4G^2} - \frac{eG}{G} \right] x_v + \left[ \frac{1}{2} N \right] \tag{18.14}
\]

\[
x_{uv} = \left[ x_u + \frac{E_v}{2E} x_u + \left( \frac{G_u}{2G} \right) u x_u v + f_u N + f u \right] \tag{18.15}
\]

\[
x_{vu} = \left[ x_u + \left( -\frac{E_v E_u}{4EG} + \left( \frac{G_u}{2G} \right) u + \frac{G_u G_u}{4G^2} - \frac{f^2}{G} \right) x_v + \left[ \frac{1}{2} U \right] \right] \tag{18.16}
\]

Because the \( x_v \) coefficient of \( x_{uu} - x_{uvu} \) is zero we get:

\[
0 = \frac{E_u G_u}{4EG} - \left( \frac{E_v}{2G} \right)_u - \frac{E_v G_v}{4G^2} + \frac{E_v E_v}{4EG} - \left( \frac{G_u}{2G} \right)_u - \frac{G_u G_u}{4G^2} - \frac{eG - f^2}{G}
\]
dividing by $E$, we have

$$\frac{eg - f^2}{EG} = \frac{E_u G_u}{4E^2 G} - \frac{1}{E} \left( \frac{E_v}{2G} \right)_v - \frac{E_v G_v}{4EG^2} + \frac{E_v E_v}{4E^2 G} - \frac{1}{E} \left( \frac{G_u}{2G} \right)_u - \frac{G_u G_u}{4EG^2}$$

Thus we have a formula for $K$ which does not make explicit use of $N$:

$$K = -\frac{1}{2\sqrt{EG}} \left( \frac{\partial}{\partial v} \left( \frac{E_v}{\sqrt{EG}} \right) + \frac{\partial}{\partial u} \left( \frac{G_u}{\sqrt{EG}} \right) \right)$$

Now we use Gauss’ theorem egregium to find an expression for $K$ in terms of $T$ of WERII

$$K = -\frac{1}{2\sqrt{EG}} \left( \frac{\partial}{\partial v} \left( \frac{E_v}{\sqrt{EG}} \right) + \frac{\partial}{\partial u} \left( \frac{G_u}{\sqrt{EG}} \right) \right) \quad (18.17)$$

$$= -\frac{1}{2E} \left( \frac{\partial}{\partial v} \left( \frac{E_v}{E} \right) + \frac{\partial}{\partial u} \left( \frac{E_u}{E} \right) \right) \quad (18.18)$$

$$= -\frac{1}{2E} \Delta (\ln E) \quad (18.19)$$

**Theorem** The Gauss curvature of the minimal surface determined by the WER II is

$$K = -\frac{4}{|T|^2(1 + u^2 + v^2)^4}$$

where $\tau = u + iv$. That of a minimal surface determined by WER I is:

$$K = \frac{4|g'|^2}{|f|^2(1 + |g|^2)^4}$$

In order to prove this thm one just sees that $E = 2|\phi|^2$ and makes use of the equation (20). Now we prove a proposition that will show WERs importance later.

**Proposition** Let $M$ be a minimal surface with isothermal parameterization $x(u, v)$. Then the Gauss map of $M$ is a conformal map.

**proof** In order to show $N$ to be conformal we only need to show $|dNp(x_u)| =$
\[ \rho(u, v)|u|, |dNp(x_v)| = \rho(u, v)|v| \] and \[ dNp(x_u).dNp(x_v) = \rho^2 x_u.x_v \] Latter is trivial because of the isothermal coordinates. We have the following eqns for \( dNp(x_u) \) and \( dNp(x_v) \)

\[
\begin{align*}
dNp(x_u) &= N_u = -\frac{e}{E} x_u - \frac{f}{G} x_v \\
dNp(x_v) &= N_v = -\frac{f}{E} x_u - \frac{g}{G} x_v
\end{align*}
\] (18.20) (18.21)

By minimality we have \( e + g = 0 \). Using above eqns the Gauss map is conformal with scaling factor \( \sqrt{\frac{e^2 + f^2}{E}} = \sqrt{|K|} \) It turns out that having a conformal Gauss map almost characterizes minimal surfaces:

**Proposition** Let \( M \) be a surface parameterized by \( x(u, v) \) whose Gauss map \( N : M \to S^2 \) is conformal. Then either \( M \) is (part of) sphere or \( M \) is a minimal surface.

**proof** We assume that the surface is given by an orthogonal parameterization (\( F = 0 \)) Since \( x_u.x_v = 0 \) by conformality of \( N N_u.N_v = 0 \) using the formulas (13) (14) one gets \( f(Ge + Eg) = 0 \) therefore either \( e = 0 \) (at every point) or \( Ge + eG = 0 \) (everywhere). The latter is minimal surface equality. If the surface is not minimal then \( f = 0 \). Now use \( f = 0 \), conformality and (13), (14) to get

\[
\frac{e^2}{E} = N_u.N_u = \rho^2 E, \quad \frac{g^2}{G} = N_v.N_v = \rho^2 G
\]

Multiplying across each equation produces

\[
\frac{e^2}{E^2} = \frac{g^2}{G^2} \Rightarrow \frac{e}{G} = \pm \frac{g}{G}
\]

The last equation with minus sign on LHS is minimal surface equation so we may just consider the case \( e/E = g/G = k \). Together with \( f = 0 \) we have \( N_u = -k x_u \) and \( N_v = -k x_v \) this shows that \( x_u \) and \( x_v \) are eigenvectors of the differential of the Gauss map with the same eigenvalue. Therefore any
point on $M$ is an umbilical point. The only surface satisfying this property is sphere so were done.

**Steographic Projection:** $St : S^2 - N \rightarrow R^2$ is given by $St(x, y, z) = (x/(1-z), y/(1-z), 0)$ We can consider the Gauss map as a mapping from the surface to $C \cup \infty$ by taking its composite with steographic projection. Note that the resulting map is still conformal since both of Gauss map and Steographic are conformal. Now we state a thm which shows that WER can actually be attained naturally:

**Theorem** Let $M$ be a minimal surface with isothermal parameterization $x(u, v)$ and WER $(f, g)$. Then the Gauss map of $M$, $G : M \rightarrow C \cup \infty$ can be identified with the meromorphic function $g$.

**proof** Recall that

$$
\phi^1 = \frac{1}{2} f(1 - g^2), \phi^2 = i2f(1 + g^2), \phi^3 = fg
$$

We will describe the Gauss map in terms of $\phi^1, \phi^2$ and $\phi^3$.

$$
\begin{align*}
x_u \times x_v &= ((x_u \times x_v)^1, (x_u \times x_v)^2, (x_u \times x_v)^3) \\
&= (x_u^2x_v^3 - x_u^3x_v^2, x_u^3x_v^1 - x_u^1x_v^3, x_u^1x_v^2 - x_u^2x_v^1)
\end{align*}
$$

and consider the first component $x_u^2x_v^3 - x_u^3x_v^2$ we have

$$
x_u^2x_v^3 - x_u^3x_v^2 = 4Im(\phi^2\phi^3)
$$

Similarly $(x_u \times x_v)^2 = 4Im(\phi^2\phi^1)$ and $(x_u \times x_v)^3 = 4Im(\phi^1\phi^2)$ Hence we obtain

$$
x_u \times x_v = 4Im(\phi^2\phi^3, \phi^3\phi^1, \phi^1\phi^2) = 2Im(\phi \times \phi)
$$

Now since $x(u, v)$ is isothermal $|x_u \times x_v| = |x_u||x_v| = E = 2|\phi|^2$. Therefore we have

$$
N = \frac{x_u \times x_v}{|x_u \times x_v|} = \frac{\phi \times \phi}{|\phi|^2}
$$
Now

\[ G(u, v) = St(N(u, v)) \]  \hspace{1cm} (18.24)
\[ = St\left( \frac{x_u \times x_v}{|x_u \times x_v|} \right) \]  \hspace{1cm} (18.25)
\[ = St\left( \frac{\phi \times \bar{\phi}}{|\phi|^2} \right) \]  \hspace{1cm} (18.26)
\[ = St(2Im(\phi^2\overline{\phi^3}, \phi^3\overline{\phi^1}, \phi^1\overline{\phi^2})|\phi|^2) \]  \hspace{1cm} (18.27)
\[ = \left( \frac{2Im(\phi^2\overline{\phi^3})}{|\phi|^2 - 2Im(\phi^1\overline{\phi^2})}, \frac{2Im(\phi^3\overline{\phi^1})}{|\phi|^2 - 2Im(\phi^1\overline{\phi^2})}, 0 \right) \]  \hspace{1cm} (18.28)

Identifying \((x, y)\) in \(R^2\) with \(x + iy \in C\) allows us to write

\[ G(u, v) = \frac{2Im(\phi^2\overline{\phi^3}) + 2iIm(\phi^3\overline{\phi^1})}{|\phi|^2 - 2Im(\phi^1\overline{\phi^2})} \]

Now its simple algebra to show that

\[ G(u, v) = \frac{\phi^3}{\phi^1 - i\phi^2} \]

But that equals to \(g\) so were done.