$T$ is a first order stable theory. Assume it has QE in $\mathcal{L}$.

Define $\mathcal{L}_o = \mathcal{L} \cup \exists \sigma \exists x \sigma$ $\sigma$ is a unary function symbol.

Let $T_o = T \cup \exists \sigma$ $\sigma$ is an automorphism $\exists x \forall x y(x) \iff y(o(x))$.

Open Problem: Does $T_o$ have a model companion?

$(T, T')$ are companions if every model of one embeds in a model of the other ($\models T \subseteq T'$).

$T'$ is a model companion of $T$ if they are companions, and in addition $T'$ is model complete.

(e.g., if $T$ has QE, $T$ is model complete. $T = Th(\mathbb{R})$

$T$ is model complete $\iff$ every formula is equivalent to an existential formula.

If $T$ is a universal theory, then its model companion is the theory of the class of existentially closed models of $T$, provided its an elementary class.

If $T$ does not have a model companion, one can still define $\Delta = \exists \text{ existential formulas } \exists x \forall x y(x) \iff y(o(x))$.

$\iff T$ is positive Robinson w.r.t $\Delta$ has universal domain for class of existentially closed models.
Assume: $T_A$ does have a model companion $T_A$. 

We know that every formula $\varphi(x) \in L_0$, $\forall x \in M$ 

$T_A + \forall x [ \varphi(x) \rightarrow \exists y \psi_q(x,y)]$ where $\psi_q$ is q.f.

If $(M, \sigma) \models T_A$ is sufficiently saturated, every possible extension of $\sigma$ (on something small w.r.t saturation) is already realised in $M$.

\text{PAPA} = "\text{Propriété d'amalgamation de paires d'automorphismes}" 

"A concept, a way of life."

\text{PAPA over models:} \quad \text{(and some for alg closed sets)}

Assume $N_1, N_2 \models T_A$ st. $M \subseteq N_1 \trianglelefteq N_2$

Assume moreover that $\sigma \in \text{Aut}(M)$

$\sigma_1 \in \text{Aut}(N_1)$ extending $\sigma$

$\sigma_2 \in \text{Aut}(N_2)$ extending $\sigma$.

$(i \in (M, \sigma), (N_1, \sigma_1) \models T_A \quad (M, \sigma) \leq (N_1, \sigma_1) \quad \text{follows by AE})$

Then $\exists (P, \bar{\sigma}) \models T_A$ and embeddings

$\rightarrow (N_1, \sigma_1) \rightarrow (P, \bar{\sigma}) \rightarrow (N_2, \sigma_2)$

\text{(Stable theories have PAPAs over models).}
Theorem: \( T \text{ stable} \implies T \text{ has PIP PA over models.} \)

Proof: Embed \( M, N_1, N_2 \) in a very (strongly) homogeneous model \( P \models T \) st. \( M \subseteq N_i \) and \( N_1 \not\subseteq N_2 \).

Let \( \bar{a} \in N_1, \bar{b} \in N_2 \). Then claim \( \sigma_1(\bar{a}) \sigma_2(\bar{b}) \equiv \bar{a} \bar{b} \).

Why? \( \text{tp}(\bar{a}/M) \) is strong, since \( \bar{a} \not\subseteq M \).

\( \text{tp}(\bar{a}/M) \) determines \( \text{tp}(\bar{a}/M\bar{b}) \) by stationarity.

Let \( \sigma \) extend \( \sigma_2 \) to an automorphism of \( P \).

So \( \sigma_1(\bar{a}) \sigma_2(\bar{b}) \equiv \sigma^{-1} \left( \sigma_1(\bar{a}) \sigma_2(\bar{b}) \right) \).

\[ \sigma^{-1} \mid_0 \sigma_1(\bar{a}), \bar{b} \]

\[ \bar{a} \bar{b} \equiv \sigma_1(\bar{a} \bar{b}) = \sigma^{-1}(\sigma_1(\bar{a})) \sigma_2(\bar{b}). \]

All that's left to show is: \( \sigma_1(\bar{a}) \equiv \sigma_1(\bar{a}) \).

So we have \( \bar{a} \not\subseteq M \Rightarrow \sigma \bar{a} \not\subseteq \sigma \bar{b} \equiv \sigma_2 \bar{b}. \) \( \tag{1} \)

Also know \( \sigma_1 \bar{a} \not\subseteq M \sigma_2 \bar{b} \) since \( N_1 \not\subseteq N_2 \). \( \tag{2} \)

Moreover \( \sigma_1 \bar{a} \sigma_2 M \equiv \bar{a} M \equiv \sigma_1 \bar{a} \sigma_2 M \) \( \tag{3} \)
\[ \Rightarrow \sigma_1 \overline{a} \equiv \sigma_1' \overline{a}. \]

Since this is a strong type, \[ \sigma_1 \overline{a} \equiv \sigma_2' \overline{a}. \]

so we now have \[ \sigma_1 \overline{a} \sigma_2 \overline{b} \equiv \overline{a} \overline{b}. \]

Conclusion: \[ \sigma_1 \cup \sigma_2 \] is a partial aut. of \( P \) and so extends to an \( \text{Aut} \ \overline{e} \).

**Theorem':** \( T \) is stable if \( \text{the PAP} \) over \( \text{closed sets} \).

**Proof:** same.

**Theorem'':** \( T \) a stable CAT has PAP over \( \text{saturated models} \).

**Proof:** same.

**Lemma:**

**Defn.** Let \[ \Phi = \{ \psi(x,y) \in \mathcal{L} / \psi \text{ is algebraic over } x \} \]

\[ \Phi_\sigma = \{ \psi(\sigma^m(x_0), \sigma^m(x_1), \ldots, \sigma^m(y_0), \sigma^m(y_1), \ldots) : \psi(x, y) \in \Phi \} \]

Assume \((M, N) \models T_n\), \( \overline{a} \in M \), \( \overline{b} \in N \) and also that

\[ \forall \psi(x,y) \in \Phi_\sigma, \text{ if } M \models \exists y \psi(\overline{a},y) \text{ then } N \models \exists y \psi(\overline{b},y). \]
Then there exists an isomorphism \( f : \text{acl}^\sigma_{\sigma} (\bar{a}) \) and commutes with \( \sigma \).

**Proof exercise**  Take \( \sigma \)-diagram & embed & see.

**Corollary**  Under the assumptions, \( \bar{a} \equiv \bar{b} \).

**Proof**

\[
\begin{align*}
M & \quad \quad \quad N \\
\text{acl}^\sigma_{\sigma} \bar{a} & \quad \xrightarrow{\text{by \ parallel}} \quad \text{acl}^\sigma_{\sigma} \bar{b} \\
\Rightarrow & \quad \quad M, N \subseteq P \\
\Rightarrow & \quad \quad \text{tp}^M \bar{a} = \text{tp}^P \bar{a} = \text{tp}^P \bar{b} = \text{tp}^M \bar{b} \\
\end{align*}
\]

In particular: if \( \forall y (x, y) \in \Phi_\sigma \), we have \( \exists y (x, y) \in \Phi_\sigma \), then \( \bar{a} \equiv \bar{b} \). It follows: every formula equivalent to a boolean combination of \( \forall \) of the form \( \exists y (x, y) \), \( r \in \Phi_\sigma \).

**Exercise:** How to express \( \forall y (x, y) \) as \( \exists z \psi (x, z) \) with \( \psi \in \Phi_\sigma \) as well. Remember \( \exists y (x, y) \) has a most \( n \) conjuncts \( / x \).
Lemma 1. Bounded type-definable sets of hyperimaginaries have hyperimaginary "codes" (canonical parameters).

Namely if \( p(x) \) is a partial type with parameters \( a \), and if \( B = \exists b : pp(x,a) \subseteq \bar{b} \) is bounded, then there exists \( c \) s.t. an automorphism fixes \( c \) iff it fixes \( B \) setwise.

Proof. Let \( B = \exists b : i < \lambda \exists b_i \) be an enumeration of \( B \).

Let \( r(\bar{x}, y) = \bar{x}p(b_i, a) \). Let \( E(y, y') = \left( \exists \bar{x} r(\bar{x}, y) \land r(\bar{x}, y') \right) \lor y = y' \).

Then \( E \) is a type-definable equivalence relation.

Also: \( a E a' \) if \( B = \)

Enumerate all formulas \( \psi(x, \bar{a}) \lor x \neq x' \) (i.e. \( T \vdash \forall x \exists y \psi(x, y) \)). Enumerate them as \( \exists i \exists x x' : i < \lambda \exists \).

For every \( i \in \lambda \) and \( x_i < a \) s.t.:

1. \( \exists x_j, j < n_i \) s.t. \( \forall x_j : \exists x' : j < i \land \psi(x_j, x') \lor x \neq x' \).

2. \( T \vdash \) "\( n_i \) is not a \( n_i \) " \( n_i \) " \( n_i \) " \( n_i \) "
(Since with α this is inconsistent, so let α be maximal such that it is.)

\[ E(y, y') = (y = y') \lor (\bigwedge_{i < \lambda} \exists x_i \neq x_j \lor \bigwedge_{j < n_i} p(x_j, y) \land \bigwedge_{j \in \mathcal{S}_n, \mathcal{C}, k} p(x_j, y') \land y, y' = tp(\alpha) ) \]

Clearly: if $\not\vdash \alpha \models tp(\alpha)$ and $B = \exists b : p(b, \alpha) \models$, then $\vdash \alpha \models E\alpha$.

Now prove converse.

Conversely, assume $\vdash \alpha \models E\alpha$. so $\not\vdash \alpha \models tp(\alpha)$.

100 cost of proof later.

**Lemma 2.** Every hyperimaginaries is interdefinable with a tuple of 1 small hyperimaginaries, where small means a quotient of a tuple of length $\leq |\Gamma|$.

(Proved in an earlier lecture.)
Let $T$ be stable (not necessarily f.o.), $M$ is a $\mathcal{L}T$-saturated model $A \models \sigma \in \text{Aut}(M)$. Assume $A, B, C \subseteq M$, independent over $M$ (i.e. $M \not\models \sigma$).

Moreover, we have $\sigma_A \in \text{Aut}(A)$ extending $\sigma$, $\sigma_B \in \text{Aut}(B)$ extending $\sigma$, $\sigma_C \in \text{Aut}(C)$ extending $\sigma$.

Finally, we have $\sigma_{AB} \in \text{Aut}(\text{bdd}(AB))$ extending $\sigma_A \cup \sigma_B$.

Then $\sigma_{AB} \cup \sigma_A \cup \sigma_{AC}$ is elementary (i.e. preserves the logic).

**Proof.** Each of $\sigma_{AB} \cup \sigma_{AC}$, $\sigma_{AB} \cup \sigma_C$, $\sigma_B \cup \sigma_{AC}$ is elementary.

Since $B$ is bdd-closed and $A \upharpoonright C$, $\sigma_B$ (from $\sigma_A$ from above).

Claim: $\text{del}(\text{bdd}(AB) \cup \text{bdd}(BC)) \cap \text{bdd}(BC) = \text{del}(BC)$.

Proof of claim is clear.

Assume $x \in \sigma_A \cap \sigma_B$. Assume $\sigma_A$ is a small hypersubmodel.

Then $\exists a \in A$, $b \in B$, $c \in C$, $p \in \text{bdd}(ab)$, $x \in \text{bdd}(ac)$, s.t. $x \in \text{del}(p, \overline{x}) \cup \text{bdd}(bc)$ and we may take them to be small.
If $x \in \text{bdd } (BC)$, let $q = tp(x/BC)$, then for every $\forall (x, x')$ contradicting $x = x'$, the type $\bigwedge_{i < \omega} q(x_i) \land \bigwedge_{i < j < \omega} q(x_i, x_j)$ is contradictory, and one only needs finitely many parameters in $BC$ for that.

Since $A \boxslash BC$ and $M$ is \(IT\)-saturated, then $M \models \forall \omega \exists x \in \text{dcl}(a, b, c)$ so $\exists a' \in M$ s.t. $a' \equiv_a \forall x, b, c$

i.e. $\exists \beta', \gamma' \in \text{bdd } (a', b), \gamma' \in \text{bdd } (a', c)$

$\exists x \in \text{dcl } (\beta', \gamma')$. \(\text{bdd } (b) = b\).

So $x \in \text{dcl } (BC)$.

Now let $d \in \text{bdd } (BC)$. I claim that $tp(d/BC) \vdash tp(d/\text{bdd } (AB) \cup \text{bdd } (AH))$.

Proof of claim: 

Let $e$ be a code for the set of $\text{bdd } (AB) \cup \text{bdd } (BC)$-conjugates of $d$.

Then on the one hand, $e$ codes a set of elements in $\text{bdd } (BC) = e \in \text{bdd } (BC)$. 
On the other hand: \( e \in \text{decl}(\text{bdd}(AB) \cup \text{bdd}(AC)) \).

\[ \Rightarrow e \in \text{decl}(BC) \text{ by claim.} \]

\[ \Rightarrow e \text{ what we wanted: if } d' \equiv d \Rightarrow d' \equiv d \Rightarrow d' \in \text{set that } e \text{ codes.} \]

Now we have \( d \in \text{bdd}(BC) \), \( e \in \text{bdd}(AB) \), \( f \in \text{bdd}(AC) \) we want to show: \( d \equiv \sigma_{BC}(d) \sigma_{AB}(e) \sigma_{AC}(f) \).

We said we know that \( \sigma_{AB} \cup \sigma_{AC} \) is elementary and therefore extends to an automorphism \( \sigma' \).

Let \( \sigma'' \) be \( \sigma'^{-1} \sigma_{BC} \).

Reduced to: \( d \equiv \sigma''(d) ef \), i.e. \( d \equiv \sigma''(d) \).

But \( \sigma' \geq \sigma_B \cup \sigma_C \Rightarrow \sigma'' \mid_{BC} = \text{id} \Rightarrow d \equiv \sigma''(d) \)

by hypothesis \( d \equiv \text{bdd}(AB) \cup \text{bdd}(AC) \)

\( \Rightarrow \sigma''(d) \Rightarrow d \equiv \sigma''(d) \). \( \square \)

Where this leads:

"Knowing" \( (\text{bdd}(AB), \sigma_{AB}) \) \( \equiv \) knowing \( \text{tp}(AB) \) in the sense of \( \text{tp} \) (where \( \sigma_{AB} = \sigma_{\text{bdd}(AB)} \)).
More generally, \( \mathcal{T}^A(a) = \text{automorphism type of} \ (\text{bdd} (\sigma^2(a)), \sigma) \).

\[
\text{bdd}_\sigma(c) = \text{bdd}^{T_A}(c).
\]

Want to define \( c \lesssim b \) if \( \text{bdd}_\sigma(a) \downarrow \text{bdd}_\sigma(b) \) \( \text{bdd}_\sigma(c) \)

Then assume \( c \lesssim_b \) & have \( c_1 \lesssim_{M_1} a \) & \( c_2 \lesssim_{M_2} b \) &

\( c_1 \equiv_{M_1} c_2 \).

Write \( A := \text{bdd}_\sigma(a, M) \) etc. so \( A \downarrow B, c_1 \downarrow_{M_1} A, c_2 \downarrow_{M_2} B \).

Then we find a new \( C \) st. \( C \downarrow AB \)

\( \Downarrow \quad \) \( c \equiv_{T_A}^{T_B} c_1 \) & \( c \equiv_{T_B} c_2 \)

so \( c \lesssim_{M} AB \) & proved ind. thm. for \( M \).