Lovely Pairs

Pair: \((M, N)\) where \(M \preceq N\) (FO)

Another way of writing it: \((M, P)\) where \(P\) is a new unary predicate \& \(P(M) = N\).

We even allow \((A, P)\) where \(A \preceq M\) and \(P(A)\) is relatively algebraically closed in \(A\).

Def: Fix a simple theory \(T\).

A pair \((M, P)\) is \(K\)-lovely where \(K \geq |T|\) if

\[\forall A \preceq M \text{ st. } |A| < K \text{ and } \forall p \in S(A) \text{ (in the sense of } T)\]

1. \(\exists a \in M \text{ a } \models P \text{ and } a \nmodels P(M)\)

2. If moreover \(p \in \text{ nond} / P(A)\) then \(\exists a' \in P(M) \text{ a' } \models P\).

Lovely is \(|T|^+\)-lovely.

We call 1 the extension property: every \(p \in S(A)\) has a nondividing extension to \(A\cup P(M)\) realised in \(M\).

We call 2 the co-heir property: it says if \(p \in S(M)\) which does not divide \(/P(M)\) then every small part of \(p\) is realised in \(P(M)\).

Note: if \((M, P)\) is \(K\)-lovely then both \(M, P(M)\) are \(K\)-saturated models of \(T\).
Fact: \(k\)-lovely pairs exist for arbitrarily big \(k\).

**Defn:** \(\langle M, P \rangle\) be a pair and \(A \subseteq M\).

\(A\) is free if \(A \not\subseteq P(M)\)\(\bigcup_{P(A)}\).

An embedding of pairs is free if it respects \(P\) and the image is free.

**Lemma** Assume \(\langle M, P \rangle\), \(\langle N, P \rangle\) are lovely pairs of \(T\), \(A \subseteq M\) is free, \(|A| \leq |T|\), \(B \subseteq N\) is free, \(|B| \leq |T|\), and \(\exists f: A \rightarrow B\) preserving \(T\)-types and \(P\).

Then \(\forall c \in M\) \(\exists A' \supseteq A\), \(B' \supseteq B\) st. same holds for \(A', B'\) via \(f'\) extending \(f\).

(Back & forth but not to because of freeness)

**Proof:** Case I: \(c \in P(M)\).

Then \(A \not\subseteq c\). Define \(A' = A \cup c\), \(P(A') = P(A) \cup c\).

So \(A'\) is free.

Final \(d \in N\) st. \(d \supseteq cA\). Then \(d \not\subseteq B\) so we may choose \(d \not\in P(N)\) by (their prop).
Case II: \( C \not\in \text{P}(M) = \text{bdd}(\text{P}(M)) \). \( \text{F}^+ \)-set model so \( \text{bdd}-\text{closed} \)

Find \( G \subseteq \text{P}(M) \) s.t. \( |C| \leq |T| \) and \( Ac \cup \text{P}(M) \) (local cho)

\text{WMIA} \ G \supseteq \text{P}(A) \).

Let \( A' := AGc \) so \( \text{P}(A') = G \)

By case I, find \( D \subseteq \text{P}(N) \) and \( f' : AG \rightarrow BD \)

Find \( d \in M \) s.t. \( \text{AG}' \cap Dd = \emptyset \) (via \( f' \))

We may choose \( d \) s.t. \( d \not\subseteq \text{P}(N) \) by each prop.

Since \( \text{P}(AG') = G' \), we have \( \text{P}(BD) = D \).

so \( B \parallel \text{P}(N) \Rightarrow BD \parallel \text{P}(N) \)

Set \( B' = BDDd : B' \parallel \text{P}(N) \)

Left to prove \( d \not\subseteq \text{P}(N) \): if \( d \not\subseteq \text{P}(N) \) then \( d \not\parallel D \)

\( \Rightarrow d \not\in \text{bdd}(D) \Rightarrow c \not\in \text{bdd}(G) \subseteq \text{P}(M) \) contradiction.

\( \square \)

For now, assume \( T \) is a complete f.o. simple theory with \( \text{AE} \).

Let \( \text{dp} = \text{L} \cup P \).

Then: if \( (M, P) \) and \( (N, P) \) are i.

lovely \( T \)-pairs then \( \text{Th}_{\text{dp}} (M, P) = \text{Th}_{\text{dp}} (N, P) \).
start a back and forth between \((M, P)\) and \((N, P)\) from \(\emptyset \equiv \emptyset\).
Moreover: if \(A \subseteq (M, P)\) is free then \(tp^{(M, P)}_{dp}(A)\) is determined by \(\delta \cdot tp^{M \setminus \{A\}}_P(A)\) and the trace of \(P\) on \(A\).

Define \(T_p := Th_{dp}(\text{lovely pairs})\). \(T_p\) is complete.

**Lemma** Let \((A, P)\) be a pair. Then it embeds freely in a \((\kappa + 1)\)-lovely pair \((\forall \kappa)\).

**Proof** Let \((M, P)\) be a \(\kappa\)-lovely pair.

First embed \(P(A)\) in \(P(M)\).

\[\text{Realise } tp(A \setminus P(A)) \text{ in } M \text{ st. } A \not\models P(M)\]

Every model of \(T_p\) is a pair, and therefore can be embedded freely in a lovely pair.

Moreover, it is easy to verify: if \((M, P) \not\subseteq (N, P) \models T_p\) then \(M\) is free in \((N, P)\).

**Converse?** True if \((M, P), (N, P)\) are lovely. (If \((M, P) \not\subseteq (N, P)\) and \(A \subseteq M\) is free then it is free in \(N\).

\[M \not\models P(N) \Rightarrow A \not\models P(N) \Rightarrow A \not\models P(N)\]
The Big Theorem: TFAE (for T):

1. Every free extension of models of Tp is elementary
2. Every model of Tp embeds elementarily in a lovely pair.
3. Every $\mathcal{L}$-lovely pair is $\mathcal{L}$-saturated as an $\mathcal{L}_p$-structure.
4. There exists a lovely pair that is $|T|^+ |-|$ saturated as an $\mathcal{L}_p$-structure.

Proof

1 $\Rightarrow$ 2: Since every pair embeds freely in a lovely pair & assumption.

2 $\Rightarrow$ 3: Let $(M, P)$ be $\mathcal{L}$-lovely, let $A \subseteq M$ s.t. $|A| < \kappa$.

Let $(N, P) \succeq (M, P)$ at $N$. We want to show $\exists a \in M$

s.t. $a \equiv^\mathcal{L}_P a^1$.

By 2 we may assume that $(N, P)$ is a lovely pair.

(Replace by all $\mathcal{L}$-simple)

Enlarging $A$ but keeping $|A| < \kappa$ we may assume $A$ is free in $M$ and therefore in $N$ (same arg as for $C$).
Now we get \( A \downarrow_{PLA} P(N) \Rightarrow A \downarrow_{PLA} C \) (cohen)

\( \Rightarrow \exists C' \subseteq P(M) \text{ st. } C' \equiv^A C. \)

Now \( \exists a' \in M \text{ st. } a C \equiv^A a' C' \) and \( a' \downarrow_{PLA} P(M) \) (extn)

Then \( P(a AC) = P(A) C \) and \( P(a' AC') = P(A) C' \)

(so traces are the same)

and \( a AC \downarrow_{PLA} P(N) \) (since \( a A \downarrow_{PLA} P(N) \))

and \( A \downarrow_{PLA} P(M) \) and \( \Rightarrow a' AC' \downarrow_{PLA} P(M). \)

ie \( ACa \) is free in \( N \) and \( AC' a' \) is free in \( M. \)

\( \Rightarrow \ ACa \equiv^A AC'a' \Rightarrow a' \equiv^A a' \)

\( \Rightarrow 4 \) by existence.

\( 4 \Rightarrow 1 \) next time.

\( 5/12. \) From last time: \( (M, P) \) and \( A \subseteq M \) then \( A \) is free if \( A \downarrow_{PLA} P(M) \)

If \( (M, P) \) lovely and \( A \subseteq M \) is free then \( tp^A(A) + P(A) \) determine \( tp^A(A). \)

\( 5/12. \) A free extension of lovely pairs is elementary.
continuing proof from last time:

\( (4) \Rightarrow (1) : \) Let \((M, P) \leq (N, P)\) be free, i.e. \(M \nsubseteq P(N)\) and 

\[ M \cup P(N) \]

It suffices to prove that \(V(M', P) \leq (M, P)\)

st. \( |M'| \subseteq |T| \Rightarrow (M', P) \leq (N, P)\) (by Loomis-Hein-Skelan).

So we may assume \( |(M, P)| \leq |T| \)

\[ (M', P) \leq (M, P) \Rightarrow M' \cup P(M) \subseteq M' \cup P(N) \]

\(P(M')\)

Since \( (K, P) \) is \( T^t \)-saturated, call it \( K^t \)

and since \( T^t \) is complete: we may assume \((K, P) \leq (K', P)\)

Finally we may assume \((K', P)\) is \( |K|^t \)-lovely.

1. \( (M, P) \leq (K', P) \Rightarrow M \cup P(K) \)

\(P(M)\)

Since \((N, P)\) is \( |K|^t \)-lovely, we may realise

\( tp(P(K)/M) \) inside \( P(N) \) (cohen prop.)

Call the realisation \( P(K) \).

Now we may realise \( tp(K/MUP(K)) \) in \( N \) st.

\[ K \cup P(N) \]

\(MUP(K)\)

\(\times\) (cohen prop.)
so \( P(M) \trianglelefteq P(K) \trianglelefteq P(N) \).

\[
\begin{array}{c}
M \upharpoonright P(N) \Rightarrow M \upharpoonright P(N) \\
P(M) & P(K)
\end{array}
\]

\( \Rightarrow K \upharpoonright P(N) \) by \( \star \).

But \( K, N \) are lovely : \((M, P) \trianglelefteq (K, P) \trianglelefteq (N, P) \). \( \square \)

Viewing this theorem:

Good: The saturated models of \( T_P \) are precisely the only lovely pairs.

Being a lovely pair is "first order".

Bad: The class of lovely pairs is not "first order."

Theorem: There always exists a cat \( T_P \) whose saturated models are the lovely pairs.

In fact, we don't need to assume that \( T \) is f.o.:

this works for every simple thick \( T \).

Good: \( T_P \) is f.o.

Bad: \( T_P \) is a non-f.o. cat. (not too bad)
Assume \( a^c (M, P) \models T_p \). [If you want, assume \( t_o \).]

Define \( a^c := CL(a^c/P(M)) \) does not depend on \( M \):
- \( M \) is free in \( N \iff a^c \upharpoonright P(N) \upharpoonright P(M) \)
- So canonical boxes are the same.

Claim: \( a^c \in \text{der}_P(a) \) is an automorphism fixing \( a \) \text{ pointwise}, \( a \text{ actswise fixes } CL(a^c/P(M)) \).

So \( tp^p(a) \) determines \( tp^p(a, a^c) \) and therefore \( tp^p(a, a^c) \).

Note: here \( \text{ der}_P, tp^p \), \( a \), mean in the sense of \( T_p \).

On the other hand, \( a \) is free: \( a^c \upharpoonright P \models a^c \upharpoonright P \).

So \( tp(a^c) \) determines \( tp(a^c) \) and therefore \( tp(a) \).

(cheating since my certain hyperimaginay, but still works)

\[ \therefore \; tp^p(a) \models tp(a) \]

If \( T_p \) is \( t_o \) i: Assume \( \exists (a, y) \models \psi \) and \( a^c (M, P) \models T_p \).

Then: \( M \models \exists y \models \psi (a, y) \iff (a, y) \models \psi \) and \( /P(M) \) (by choice)

Exercise
\[ \models \psi (a, y) \text{ and } /a^c \]
Fact. $T_p$ admits QE. up to boolean combinations of $\forall x \in P \exists y (x, y)$.

Sketch of Proof. Assume $a, b$ both satisfy some formulas of this kind.

Let $tp_a(a) := \exists x \in P \exists y (x, y) : \forall z \in P \exists y (z, y) \neg \exists x \in P \exists y (x, y) : \forall z \in P \exists y (z, y) \neg$

Assume that $a \in (M, P)$, $b \in (N, P)$, $tp_a(a) = tp_b(b)$.

Then $q(x) \mid U$ is a copy of $P(M) \subseteq P \setminus U$

\exists a copy of $P(N) \subseteq P \setminus U$ is consistent

$\models tp(a/p(M)) \land \models tp(b/p(N))$

an $\mathcal{P}$-type with constants for $P(M), P(N)$.

$c_t : t \in P(M), d_s : s \in P(N)$.

$q(x) \land P(c_t) \land P(d_s) \land x \not\models tp(a, p(M)) \land x \not\models tp(b, p(N))$

Finitely realizable: $\varphi(x, c)$ in $P(M)$ and $\varphi(x, d)$ in $P(N)$, \therefore $q \models \exists y, z \in P \exists y (x, y) \land \varphi(x, z)$ consistent.
Since \( a \) satisfies the negative part of \( q \):

\[
\text{tp}(\tilde{a} / P(K)) \text{ is a cohen of } \text{tp}(\tilde{a} / P(M)), \text{tp}(\tilde{a} / P(N))
\]

\[
\tilde{a} \not\in P(M) & \tilde{a} \not\in P(K)
\]

\[
cb(\tilde{a} / P(M)) = cb(\tilde{a} / P(K)) = cb(\tilde{a} / P(N)) = a^c
\]

so \( a, a^c = \tilde{a}, \tilde{a}^c = b, b^c \)

so \( a \equiv b \implies \text{QE} \).

Theorem. Let \( a, b, c \in (M, P) \models T_p \).

Let \((a_i), b_i, c_i : i < \omega \) be a Morley seq. for \( a, b, c / P(M) \) (in sense of \( T \)).

Then TFAE: \( 1. \ a \not\in b \) and \( \tilde{a} c \not\in (bc)^c \)

\( \tilde{a} \) doesn't matter what \( M \) is again.

\( 2. \ a \not\in b \) and \( \tilde{a} c \not\in b c \)

\( 3. \ (a_i : i < \omega) \not\in (b_i : i < \omega)

(\( c_i : i < \omega \))

Call these notions: \( a \upharpoonright^c b \)

\( \tilde{a} \)
So it follows immediately from (3) that \( \mathcal{U} \) satisfies all axioms for independence except maybe independence.

**Prop.** Assume \( a, b \subseteq P \) and \( b_i \subseteq a \subseteq P \) for \( i \in \{1, 2\} \)

and \( b_1 \equiv P b_2 \). Then \( \exists b \) st. \( b \subseteq P a \), \( a \subseteq P b \), \( b_i \equiv P b_i \leq \text{bd}(c) \).

So \( T_p \) is simple and \( \mathcal{U} = \text{nondividing} \) and

\[ \text{bd}(P) = \text{dcl}(C) \text{dcl}(\text{bd}(C)). \]

We said \( a \equiv P b \Rightarrow \exists \text{aut. sending } a \text{ to } b \)

\[ \Rightarrow \exists \text{aut. sending parallelism class of } \]

\( T_p(a/p) \) to that of \( T_p(b/p) \),

\( T_p \)-types are the same thing as types of \( T \)-parallelism classes.

**Example.** \( U(ACF_p) = \omega \), \( U(\text{vector space } p) = 2 \).

But \( U(ACF) = U(\text{vector space}) = 1 \).