Second proof of remark given last time:

Assume \( x_0 \leq x_1 \leq x_2 \leq \ldots \) are increasing types of variables.

\[
p_0(x_0) \leq p_1(x_1) \leq \ldots \quad \text{are increasing complete types}
\]

Let \( a_i \vdash p_i \quad \forall i \).

For each \( i \leq j \), let \( a_{j,i} \) be the subtuple of \( a_j \) corresponding to \( x_i \).

Define \( b_0 \leq b_1 \leq b_2 \leq \ldots \) s.t. \( b_i \vdash p_i \).

Let \( b_0 = a_0 \). Assuming we have \( b_i \), the \( \text{tp}(b_i) = \text{tp}(a_{i+1,i}) = p_i \)

so there is an automorphism \( f \) sending \( a_{i+1,i} \) to \( b_i \).

Let \( b_{i+1} = f(a_{i+1,i}) \). Let \( b = \bigcup b_i \). Then \( b \vdash \bigcup p_i \).

Continued on next page...
Definition: A partial type $p(x, b)$ divides over $c$ if there is an indiscernible sequence $(b_i : i < \omega)$ in $\text{tp}(b/c)$ such that $\forall x, (x, b_i) \models U p(x, b_i)$ is consistent.

Remark 2: If $(b_i : i < \omega)$ is an indiscernible sequence in $\text{tp}(b/c)$ then it has an automorphic image which is also $c$-indiscernible.

Proof: By compactness, for every $\lambda$, there is a similar sequence $\phi$ in $\text{tp}(b/c)$. Let $p_n(x_0, \ldots, x_{n-1}) = \text{tp}(b_0 \ldots b_{n-1}) : \phi(x) \land \forall b_0 \ldots b_{n-1} \in \text{consistent}. 

Now extract a $c$-indiscernible sequence: $(b_i'' : i < \omega)$

1. $b_i'' = b_i$ for all $i$.
2. $\forall i_0 < \ldots < i_{n-1} \in \omega : p_n(b_{i_0}'' \ldots b_{i_{n-1}}'') \Rightarrow p_n(b_{i_0}'' \ldots b_{i_{n-1}}'') \Rightarrow \text{tp}(b_i : i < \omega) = \text{tp}(b_i'' : i < \omega)$. 

$b_i''$ is an automorphic image of $b_i$.

Definition: $a \nmid_b c$ (read: $a$ independent of $b$ over $c$) if $\text{tp}(a/bc)$ does not divide $a/c$. 


Proposition: \( a \perp b \iff \forall c, \text{ every indiscernible sequence in } t_p(b/c) \text{ has an automorphic image in } t_p(b/ac) \).

Proof: \( \Rightarrow \): Assume \( a \perp b \), i.e. \( t_p(a/bc) \) does not divides (and) over \( c \), while \( p(x,bc) = t_p(a/bc) \).

Let \( (b_i : i < \omega) \) be an indiscernible sequence in \( t_p(b/c) \).

Then \( (b_i : c < \omega) \) is indiscernible in \( t_p(bc/c) \).

[By the remark, there is an automorphic image \( (b_i^* : i < \omega) \) which is \( c \)-indiscernible and in \( t_p(b/c) \)]

\( \Rightarrow \) \( (b_i^* : c < \omega) \) is indiscernible in \( t_p(bc/c) \).

Since \( t_p(a/bc) \) divides over \( c \), there is \( a' \models \forall p(x,bc) \).

In particular, \( a' \models t_p(a/c) \).

Applying, let \( f \) be an \( c \)-automorphism s.t. \( f(a/c) = ac \).

Therefore \( (b_i^*) = f(b_i^*) \) is an automorphic image of \( (b_i) \)

and \( \forall p(x,b_i^*) \Rightarrow b_i^* \models t_p(b/ac) \).

\( \Leftarrow \): Let \( (b_i c) \) be any indiscernible sequence in \( t_p(b/c) \).

We need to find \( a' \models \forall p(x,bic) \).

By assumption, \( (b_i) \) has a \( c \)-automorphic image \( (b_i^*) \) in \( t_p(b/ac) \).

Let \( f \) be the \( c \)-automorphism & let \( a' = f'(a) \).
Then \( \Lambda p(a, b / c) \rightarrow \Lambda p(a', b / c) \).

**Corollary:**

1. **Downward right-hand transitivity:**
   
   \[ a, b, c, d : a \Downarrow b, d \rightarrow a \Downarrow b \wedge a \Downarrow c \wedge b \Downarrow c. \]

2. **Upward left-hand transitivity:**
   
   \[ a \Downarrow b \quad \text{and} \quad d \Downarrow b \rightarrow a \Downarrow c \quad \text{and} \quad d \Downarrow c \]

**Proof**

1. Assume \( a \Downarrow b d \) then if (b) is \( c \)-indiscernible in \( \text{tp}(b / c) \) then by extension/extension, we can find (d) st.
   
   \( (b ; d) \) is \( c \)-indiscernible in \( \text{tp}(b d / c) \).

   (Extend to \( (b ; i < \lambda) \), for each \( i \) find \( d i \) st. \( b d i \equiv_c b d \).

   and extract a \( c \)-indiscernible sequence \( (b ; d i) \).

2. \( b < w \equiv_c b < w \) (both are similar \( c \)-indiscernible seq of same length).

   So we may assume \( b < w = b < w' \).

   Since \( a \Downarrow b d \), there is a \( c \)-automorphic image \( (b ; d) \) in \( \text{tp}(b d / c) \).

   In particular \( (b ; d) \) is a \( c \)-automorphic image of \( (b) \) in

   \( \text{tp}(b d / c) \rightarrow a \Downarrow c \).

   \( (b ; d) \)

   \( (b ; d) \)

   Now: let \( (a ; b c d) \) be bi-indiscernible in \( \text{tp}(b c d / c) \).

   Then \( (a ; b c d) \) is bi-indiscernible in \( \text{tp}(b c d / c) \).

   Let \( \text{tp}(a / b c d) = q(x b c d) \).
So $\forall y (x, bc \in y) \text{ is consistent} \Rightarrow a \not\subset bc d$.

(2) Assume $a \not\subset b, d \not\subset b$.

Let $(b_1')$ be a $c$-indiscernible sequence in $tp(b/c)$.

Since $a \not\subset b$, there is a $c$-automorphic image $(b_1')$ in $tp(b/ac)$.

By recursiveness,

$(b_1')$ is $c$-indiscernible and by a previous remark has a $c$-automorphic image which is still in $tp(b'/ac)$ and $b$ in addition is ac-indiscernible.

So we may assume $(b_1')$ is ac-indiscernible.

Since $d \not\subset b$, then $(b_1')$ has an ac-automorphic image in $tp(b'/acd)$.

Conclusion $(b_1')$ has a $c$-automorphic image in $tp(b'/acd)$.

$$\exists a \in c \not\subset b$$

Lemma. A partial type $p(x, b)$ divides $e$ iff there is a formula $p(x, b') \in p(x, b)$ which does.

(Convention: all partial types are closed under conjunction)

Proof $\Leftarrow$: clear.

$\Rightarrow$: Assume $(b_1')$ is $c$-indiscernible and $e \cup p(x, bi)$ is inconsistent. By compactness, only finitely many
formulas are required for inconsistency, say
\[ \psi_0(x, b_0) \in p(x, b_0), \ldots, \psi_{k-1}(x, b_{k-1}) \in p(x, b_{k-1}). \]
Let \( \psi = \bigwedge \psi_i(x, b_i) \). Then \( \psi(x, b) \in p(x, b) \).

\[ \text{and} \quad \bigwedge \psi(x, b_i) \text{ is inconsistent} \implies \psi(x, b) \text{ divides } \langle c \rangle. \]

**Corollary**  Finite Character \( a \perp b \implies (a, b, c \text{ are possibly infinite}) \)

\[ \iff \forall a' \in a \text{ and } b' \in b \text{ finite, } a' \perp b' \]

**Proof** \[ \iff \text{clear.} \]

\[ \iff \text{if } a \perp b \text{ then there is a formula } \psi(x, b c) \in p(a/x) \]

which divides over \( c \).

Now only finite subtypes and \( x' \in x \) actually appear in \( \psi \). Let \( a' \subset a \) correspond to \( x' \in x \).

\[ \implies \psi(x', b' c) \in p(a'/b' c). \]

\[ \implies a' \perp b' \]

\[ \Box. \]