\( p(x, b) \) divides \( c \) if \( \exists \) \( c \)-indiscernible sequence \((b_i)\) in \((b, b/c)\) s.t. \( \forall x, b_i \) is inconsistent.

\( \exists K < \omega \) \( \exists \psi(x, b) \in p(x, b) \) s.t. \( \forall x, b_i \) is inconsistent \( \Rightarrow \exists \psi(x, b_i) \) is \( K \)-consistent.

If we have negations then we can say \( \forall x \neg \exists k \exists x \psi(x, b) \) and apply ext/ext to get \((b_i)\) indiscernible.

**Defn.** Let \( \psi(x, y) \) be a formula \((x, y \notin \text{tuple of variables})\) \( K < \omega \).

\( \psi(y_0, y_{k-1}) \) another formula s.t. each \( y_i \) has the same length as \( y \). [each \( y_i \) is in the sort of \( y \)].

Then \( \psi \) is a \( K \)-inconsistency witness for \( \psi \) if

\[ \Gamma \vdash \exists y \psi(y) \land \forall k \psi(x, y). \]

**Defn.** A formula \( \psi(x, b) \) divides \( c \) w.r.t. a \( K \)-inconsistency witness \( \psi^* \) if there exists a sequence \((b_i)\) in \( p(b, b/c)\) satisfying:

\[ \forall i_0, \ldots, i_{k-1} \psi(b_{i_0}, \ldots, b_{i_{k-1}}) = \psi^*(b) \quad [\psi^*(y_0, \ldots) = \forall i_0, \ldots, i_{k-1} \psi(y_{i_0}, \ldots, y_{i_{k-1}}) \text{ for all } i_0, \ldots, i_{k-1}]. \]
Prop: 1. \( \varphi(x,b) \) divides \( c \) \( \implies \) 2. divides \( c \) \( \forall \) some \( k \)-inconsistency witness \( \varphi \) \( \implies \) 3. \( \exists c \)-indiscernible sequence \((u_i)\) in \( tp(blc) \) s.t. \( \varphi(b_0, b_{k-1}) \).

Proof: 1.\( \implies \) 2. \( \exists \) an indiscernible seq \((u_i)\) in \( tp(blc) \) s.t. \( \land p(x, b_i) \) is inconsistent.

By compactness \( \exists k < \omega \) s.t. \( \land \varphi(x, y_i) \) is inconsistent.

Let \( q(y_0, \ldots, y_{k-1}) = tp(b_{2k}) \):

\( \Rightarrow \ q(y) \land \land_{i < k} p(x, y_i) \) is inconsistent.

\( \Rightarrow \ \exists \ y(\tilde{y}) \in tp(b_{2k}) \) s.t. \( \varphi(\tilde{y}) \land \land_{i < k} \varphi(x, y_i) \) is inconsistent \( \checkmark \).

2.\( \implies \) 3. We have a sequence \((u_i)\) in \( tp(blc)\) satisfying \( \tilde{\varphi} \).

Since compactness applies to \( \varphi \) (it does not apply to \( \exists \exists \forall (x, y) \)), we may apply extension/extraction to get a sequence \((u'_i)\) indiscernible \( /c \) having same properties.

3.\( \implies \) 1. Clear. \( (\land \varphi(x, b_i) \) is inconsistent because \( \forall \varphi(b_0, b_{k-1}) ) \).
Defn: Let \( \alpha \) be a type of variables.

Then

\[
\Xi(x) = \{ (y(x, y), \psi(y_0, \ldots, y_{k-1}) : \psi(x, y) \in \Delta; \ k < \omega) \}
\]

\( \psi \in \Delta \) is a \( k \)-inconsistency witness for \( \psi \).

\( \alpha \) is fixed but \( k \) and \( \psi \) vary.

Defn: For every partial type \( pl(c) \) (with parameters) we associate a "rank", written \( D(p, \Xi) \), which is a set of sequences in \( \Xi \) of ordinal length.

For \( \xi \in \Xi^\alpha \) we decide whether \( \xi \in D(p, \Xi) \) by induction on \( \alpha \):

\( \alpha = 0 \): \( \langle \cdot \rangle \in D(p, \Xi) \) iff \( p \) is consistent.

\( \alpha \) limit: \( \xi \in D(p, \Xi) \) iff \( \forall \beta < \alpha \); \( \xi |_\beta \in D(p, \Xi) \).

\( \alpha = \beta + 1 \): \( \xi = \langle \Theta, (\psi(x, y), \psi(y)) \rangle \) where \( \Theta \in \Xi^\beta \).

Assume \( p \) is over \( b \).

Then \( \xi \in D(p, \Xi) \) iff \( \exists c \) s.t. \( \psi(x, c) \) divides \( b \) wrt \( \psi \) and \( \Theta \in D(p(x, \forall \psi(x, c), \Xi) \).
Obvious things: If $\xi \in D(p, \equiv)$ and $p \vdash q$ then $\xi \in D(q, \equiv)$.

- $D(p, \equiv)$ is closed under subsequences.

Still need to get rid of $p/b$ assumption ...

**Remark** We prove by induction on $\alpha$ that for $\xi \in \equiv^\alpha$ and $p(x, b) \equiv q(x, b')$ that $\xi \in D(p(x, b), \equiv)$ iff $\xi \in D(q(x, b'), \equiv)$.

(i.e. choice of set of parameters $b$ is not important)

**Proof:** $\alpha = 0 \checkmark$

$\alpha$ limit $\checkmark$

Let $\alpha = \beta + 1$, $\xi = \langle \theta, (\psi, \psi) \rangle$ and assume $\xi \in D(p, \equiv)$.

$\Rightarrow$ $\exists c \ s.t. \psi(x, c)$ divides $b$ w.r.t. $\psi$, $\theta \in D(p \psi(x, c), \equiv)$

$\Rightarrow$ $\exists b'$-indiscernible sequence $(c_i)$ in $\mathcal{L}_p(c/b)$ s.t.

$\psi(c_0, c_{k-1})$ and $\theta \in D(p \psi(x, c), \equiv) = D(p \psi(x, c), \equiv)$

By extension/extraction there is a $b b'$-indiscernible

sequence $(c_i)$ similar over $b$ to $(c_i)$.

($\therefore \psi(c_0, c_{k-1}) \Rightarrow \psi(c_0, c_{k+1})$.)
\( \Theta \in D(p \land \psi(x, c_0'), \equiv) = \bigwedge D(q \land \psi(x, c_0'), \equiv) \)
and \( \psi(x, c_0') \) divides \( bb' \) w.r.t. \( \psi \) and thus \( bb' = \bigwedge E D(q, \equiv) \).

**Defn:** \( T \) is thick if indiscernibility is type-definable i.e.

A tuple \( x \in \) partial type \( \Theta(x, c) \) saying precisely that \( (x_c) \) is indiscernible.

**Remark:** Let \( a, b \) and \( (a_i \prec a_0) \) be possibly infinite tuples. Then \( (a_i) \) is indiscernible \( b \) iff

for finite subtypes \( b' \subseteq b \) and \( a_0' \subseteq a_0 \),
if \( a_i' \subseteq a_i \) are the corresponding subtypes,
the sequence \( (a_i' b', \prec a_0) \) is indiscernible.

It follows that for \( T \) to be thick, it suffices that indiscernibility of sequences of finite tuples be definable and we get definability of indiscernibility / something.

**Remark:** A first order theory is thick:

Let \( p(x, y) \) be a partial type, \( x \& y \) possibly infinite tuples.
Assume \( p \) is closed under finite conjunction.

Let \( q(y) = \exists x' \psi(x', y') \cdot x' \subseteq x, y' \subseteq y \) finite \( \equiv \psi(x, y) \psi x \psi y \).

Then \( q(y) = \exists x' p(x, y) \). By compactness.
clear \implies \text{compactness, if } \vdash q_i(b), \text{ then } p(x, y, b) \text{ is consistent.} \quad \Box

Let \( g \in \tau^\alpha \), i.e. \( g = ((\varphi_i(x, y), \psi_i) : i < \alpha) \).

Define \( \text{div}_{\alpha, g} (x) \) to be the partial type saying:

There exist \( c_i : i < \alpha \) of the lengths of the corresponding \( y_i \) st.

1. \( \forall x \exists y (x, c) \)

2. For all \( i < \alpha \), there exists a \( b, c, \) -indiscernible sequence \( (c_i^j : j < \omega) \) with \( c_i^0 = c_i \) and

\( \psi_i(c_i^0 \ldots c_i^{k_i-1}) \).

Prop: Let \( p(x) \) be a partial type over \( \mathcal{L} \).

Then \( g \in \text{Di}(p, \tau) \) iff \( p(x) \land \text{div}_{\alpha, g} (x) \) is consistent.

23. Proof: By induction on \( \alpha \), where \( g = ((\varphi_i, \psi_i) : i < \alpha) \).

\( \alpha = 0 \), \( \iff g \in \text{Di}(p, \tau) \) iff \( p \) is consistent

\( \iff p(x) \land \text{div}_{\alpha, g} (x) \) is consistent \( \iff p(x) \land \text{div}_{\alpha, g} (x) \) is consistent \( \Box \)
Theorem TFAE

1. \( T \) is simple (i.e., \( \kappa^0(\Gamma) < \infty \)).

2. For all \( (\psi, \psi) \in \Xi \), \( \exists \ell < \omega \) st. there is no sequence \( \{ b_i : i < \ell \} \) where each \( \psi(x, b_i) \) divides \( \prod_{i < \ell} b_i \) w.r.t. \( \psi \) and \( \bigwedge_{i < \ell} \psi(x, b_i) \) is consistent.
3. \( k^0(T) \leq |T|^t \)

4. \( \forall p \quad \text{tp}(p, \equiv) \leq \equiv/k|T|^t \).

Proof: 1 \( \Rightarrow \) 2:

Assume \( k^0(T) < \infty \) but 2 is false.

there are \( \psi_i(x, \varphi) \equiv \forall x \text{ s.t. } \forall l < \infty \exists (b_i : i < l) \)

\( \forall (x, b_i) \text{ divides } /b_i \text{ and } \forall \psi_i(x, b_i) \text{ is consistent} \)

\( \Rightarrow \) by compactness \( \exists (b_i : i < k^0(T)) \text{ s.t.} \)

\( \psi_i(x, b_i) \text{ divides w.r.t. } \psi /b_i \forall i < k^0(T) \)

and \( \forall i < k^0(T) \psi(x, b_i) \text{ consistent} \).

So let \( a \not\models \bigwedge \psi_i(x, b_i) \), then \( \text{tp}(a/b_{k^0(T)}) \)

contradicts the definition of \( k^0(T) \).

2 \( \Rightarrow \) 3 Assume \( \equiv/k|T|^t \) is false.

Then we have singleton \( s^T \) \( A \) s.t.

\( \text{tp}(a/\varphi) \text{ divides over every } A \in A \text{ s.t. } |A_0| \leq |T|^t \).

Construct a sequence \( (b_i : i < |T|^t) \text{ in } A \):

\( \forall i \exists \psi_i(x, b_i) \in \text{tp}(a/\varphi) \text{ which divides }/b_i \)
Moreover, let $\psi_i(x,bi)$ divide $b_{ci}$ w.r.t $\psi_i$.

Since $|\Xi| = |\Omega|$, there is a pair $(\psi, \psi) \in \Xi$.

$\Xi = \{ e \in (\psi, \psi) \text{ is infinite.} \}

\Rightarrow \forall e \in \Xi \psi(x,bi) \text{ div } b_{e} e_{j} : j < i \text{ w.r.t } \psi.

Contradicting (2).

(3) $\Rightarrow$ (1) by defn.

(2) $\Rightarrow$ (4) if $\exists \xi \in \Xi^{1+}, \xi \in D(p, \Xi)$, then some argument.

Then some pair $(\psi, \psi)$ appears infinitely many times in $\xi$,

contradicting (2) (look at a realisation $a \notin \text{ div } \psi, \xi$).

(4) $\Rightarrow$ (2) if $\Xi$ is true, obtained by compactness.

If (2) is false for $(\psi, \psi)$, then by compactness,

$\text{div}_{\psi}(\psi, \psi)_{1+}$ is consistent $\Rightarrow$ not (4). \qed
So from now on, assume $T$ is simple

$\Rightarrow \forall p \ D(p, \Xi) \text{ is a set, closed under limits (by def).}$

$\Rightarrow \text{contains maximal element.}$

$[\xi \subseteq \xi \text{ if } \xi \text{ is an extension of } \xi]$.

**Theorem** Let $p = tp(a/b)$ and $q = tp(a/bc)$.

**TFAE**

1. $D(p, \Xi) = D(q, \Xi)$.
2. $\exists \xi \in D(p, \Xi)$ maximal that is also in $D(q, \Xi)$ (not unique still).
3. $q$ does not divide over $b$.

**Proof**

1. $\Rightarrow$ 2. maximal elements exist.
2. $\Rightarrow$ 3. assume $q$ divides over $b$.
3. $\Rightarrow$ 1. (tricky part)

Let $\xi = \{(\psi_i, \psi_i) : i < \lambda \}$.
We prove by induction that if \( \xi \in D(p, \Xi) \) then
\[ \xi \in D(q, \Xi). \] (Converse is clear since \( p \leq q \).

To come later...

\[ \text{Cor of } \Xi \Rightarrow \Xi : \text{ Extension is true.} \]

**Proof** We are given \( a, b, c \).

Let \( \xi \in D(\text{tp}(a/c), \Xi) \) be maximal.

Since \( \text{tp}(a/c) \) is over \( b, c \) as a partial type,
\( p(x) \wedge \text{div}_{b, c}(x) \) is consistent.

Let \( a' \not\models p(x) \wedge \text{div}_{b, c}(x) \).

Then \( a' \equiv_c a \) and \( a' \not\models b \) since \( D(\text{tp}(a'/b/c), \Xi) \)
contains a maximal element of \( D(\text{tp}(a'/c), \Xi) \). \( \Box \)

Now since we have extension, we have symmetry, transitivity etc. Still have independence theorem.

**Lemma** Assume \((a_i : i \leq \omega) \) is \( c \)-indiscernible.

Then \( a_\omega \not\models c \).
Proof. Let \((c_j : j < \omega)\) be \(a_\omega\)-indiscernible.

in \(\text{tp}(c / a_\omega)\), let \(\varphi(x, a < n, c) \in \text{tp}(a_\omega/a < \omega c)\).

Then \(\vdash \bigwedge_j \varphi(a_n, a < n, c_j)\) (since \(\vdash \varphi(a_n, a < n, c)\))

\(\implies \varphi(x, a < n, c) \text{ dnd } / a < \omega\) \(\Box\)

Lemma. Let \((a_i : i < 2\omega)\) be a \(c\)-indiscernible sequence. Then \((a_{\omega + i} : i < \omega)\) is a Morley sequence over \(c, a < \omega\).

Proof. \((a_i : i < \omega)\) is \(a_\omega\)-indiscernible over \(c \cup \{ a_j : \omega \leq j < 2\omega \}\).

\(\implies a_\omega \nvdash_c a_\omega \quad \text{trans} \quad a_\omega \nvdash a_\omega c\)

Notice \(0 : \omega, \omega + 1, \dotsc \) \(\nvdash c, a_\omega, a_\omega, a_\omega, a_\omega, \dotsc\).

\(\text{tp}(a_\omega, a_{\omega + n} / c) = \text{tp}(a_\omega, a_{\omega + n} / c)\).

\(\implies a_{\omega + n} \nvdash a_{\omega + n} c\)

\(\implies a_{\omega + n} \nvdash a_{\omega + n} c\)

want to prove:

By induction, \(a_\omega \nvdash a_{\omega + n} c\) \(a_{\omega + n} c\) \(\Box\)

For \(n = 0\) \(\check{\Box}\).
For \( n+1 \) we have \( \alpha_{\omega+n} \upharpoonright \alpha_{\omega+n-1} \sim_{\text{trans}} \omega \),

\[ \iff \alpha_{\omega+n} \cup \alpha_{\omega+n-1} \sim_{\text{trans}} \alpha_{\omega^n \omega} \]

\[ \iff \alpha_{\omega^n \omega} \cup \alpha_{\omega^n \omega-1} \sim_{\text{trans}} \alpha_{\omega^n \omega} \]

So now by symmetry, \( \forall m \alpha_{\omega^m} \cup \alpha_{\omega^m-1} \sim_{\text{trans}} \alpha_{\omega^m} \). \( \square \)

**Corollary**

Assume that \( p(x, b, c) \) does not divide over \( c \).

Let \( (b_i) \) be \( c \)-indiscernible in \( \text{tp}(b/c) \).

Then \( \cup p(x, b_i, c) \) is consistent and\( \vdash P \).

[Last few lemmas: Kim "Forking in Simple Theories"]]  

**Proof** Extend \( (b_i: i < \omega) \) to a similar sequence \( (c, b_i: i < \omega) \).

By nondividing, \( \exists a \in p(x, b_0, c) \) and

\[ a \vdash p(x, b_0, c) \]

\[ \iff \exists \sigma \in \text{tp}(b_0, c) \exists x \in p(x, b_0, c) \sigma \]

\( (b_{\omega i}) \) is \( b_{\omega_1}c \)-indiscernible. \( \iff \) since \( a \vdash b_{\omega_0}c \)

We may assume that \( (b_{\omega i}: i < \omega) \) is \( b_{\omega_0}c \)-indiscernible

since we can send it to one by an \( (b_{\omega_1}, c) \)-automorphism.
But \((b_{\text{wti}} : <\omega)\) is a Morley sequence over \((b_{<\omega}, c)\).

\[=\] by a previous result, since it is also

\[a, b_{<\omega}, c -\text{indiscernible, we have } a \not\subseteq b_{<\omega}, \text{ wtii}, \ldots\]

Now add in \(a \not\subseteq b_{<\omega} \rightarrow a \not\subseteq b_{<\omega}, \text{ wtii}, \ldots\)

We also have \(\forall c \in \mathcal{C} : p(a, b_{\text{wti}}, c) \forall c <\omega\).

\[\Rightarrow \bigcup_c p(x_{\text{wti}}, c) \text{ does not divide } /c\]

\[\Rightarrow \bigcup_c p(x_{\text{wti}}, c) \text{ does not divide } /c. \quad \square\]

25. Improved Improved Extension

If \(p(x, bc)\) is a partial type over \(bc \in d/c\), then it can be extended to a complete type over \(bc\) that does not divide \(/c\).

Proof. By basic extn, \(\exists \text{ Morley sequence } (b_i)\) for \(b/c\).

Since \(p(x, bc)\) and \(c \in \mathcal{C}\) \(\mathcal{M} \vdash \forall x (p(x, bc))\)

We may assume \((b_i)\) is \(a/c -\text{indiscernible}\)

\[\Rightarrow a \not\subseteq b_i\]

Then \(q(x_{b/c}, c) := tp(a'/bc) \text{ and } /c \& q(x, bc)\) is 0

what we wanted \(\square\)