Simplicit Theory


1) \( k \)-saturated \( k \)-strongly homogeneous structure \( \mathcal{U} \), \( \Delta = \text{logic} \).

\( \mathcal{U} \) is strongly hom. by assumption.

Compactness: if \( \Sigma \) is a set of formulas finitely realisable in \( \mathcal{U} \), then \( \Sigma \) is realisable in \( \mathcal{U} \).

\( \text{Lemma: Suppose } \Delta \text{ is the positive set of formulas on predicates } \text{ and suppose } \mathcal{U} \text{ has compactness for } \mathcal{U} \text{. Then it has compactness for subsets of } \Delta. \)

\( \text{Proof: Let } \Sigma \subseteq \Delta \text{ be a set of formulas that is finitely realisable. Let } \Xi \text{ be a minimal finitely realisable subset containing } \Sigma \text{ (with some free variables).} \)

If \( \varphi, \psi \in \Xi \), we claim \( \varphi \in \Xi \) or \( \psi \in \Xi \). Otherwise we can find \( \varphi, \psi \in \Xi \) st. \( \varphi \land \psi \) not realisable, \( \varphi \lor \psi \) not realisable, but then \( \varphi \land \psi \land (\varphi \lor \psi) \) is not realisable.

So put each \( \varphi \in \Sigma \) in disjunctive normal form \( \varphi = \bigvee_i \varphi_i \).

\( \Rightarrow \text{one } \varphi_i \in \Xi \Rightarrow \text{ every conjunct of } \varphi_i \in \Xi. \)

\( \Rightarrow \exists \text{ Ri } \exists \Sigma \models \Xi \text{ implies } \Sigma \text{ finitely realisable } \Rightarrow \text{ realisable } \Rightarrow \text{ realisable.} \)

2) Hilbert Space Example.

\( \mathcal{U} = \text{unit closed ball in a large Hilbert space } \mathcal{H}. \)

\( \Delta = \Xi \cup \{ \sum \lambda_i x_i \mid \lambda_i \in \mathbb{R}, x_i \in \mathcal{H} \} \)

\( \Delta = \text{positive set of formulas in } \mathcal{H}. \)

(Rank: inner product is expressible in these terms: \( (x, y) = \frac{\|x+y\|^2 - \|x-y\|^2}{4} \).

\( (x, y) \leq 1 \Rightarrow \|x+y\|^2 \leq \|x-y\|^2 + 4r \)

\( \Rightarrow \lambda \downarrow (\|x\| \leq \|x-y\|^2 \leftrightarrow \lambda) \lambda \|x+y\|^2 \leq \frac{\|x\|^2 + 4r}{\lambda} \).
Check $W$ is a universal domain.

Homogeneous: Let $f: \mathbb{A} \to W$ be a partial homomorphism.

So for $x_i \in \mathbb{A}$, $\lambda_i \in \mathbb{R}$, we know

$\sum \lambda_i x_i = 0 \implies \sum \lambda_i f(x_i) = 0 \implies \sum \lambda_i f(x_i) = 0$

So $f$ extends to $\mathbb{A}^+ \to W^+$.

So since $\mathbb{A}^+$ is type-definable, it extends to $\overline{\mathbb{A}}^+ \to W^+$. 

$f$ preserves the metric so this is an embedding.

Since $W$ is $\omega$-saturated, we can pick an isomorphism $\mathbb{A}^+ \to f(\overline{\mathbb{A}}^+)^+$, and we get an onto $f: \mathbb{A} \to W$ restricting to $W \to W^+$.

Compactness: Start with $\Sigma$, let $\mathfrak{x}$ be the variables in $\Sigma$.

Let $W = \oplus \mathbb{R} \mathfrak{x}_i$, let $W_0 = W$ st. $W_0 = \Sigma \mathfrak{x}_i \in \Sigma \mathfrak{x}_i \mathfrak{x} \leq 1$.

Look at $[C j] W_0$ functions from $W_0$ to $[C j] W_0$ with Fréchet?

By the lemma, it's enough to consider sets of predicates.

Each predicate $\Sigma \mathfrak{x}_i \mathfrak{x}_i \mathfrak{x} \leq 1$ defines a closed subset of $[C j] W_0$.

The axioms for a norm $\|x + y\| \leq \|x\| + \|y\|$ and $\|r x\| = |r| \|x\|$.

The requirement that $\|\cdot\|$ defines a semi-positive-definite inner product defines a further closed subset. Call it $D \subseteq [C j] W_0$.

$\Sigma = \Sigma \mathfrak{p}_j \mathfrak{x}_j \mathfrak{x}_j \mathfrak{x} \leq 1$ defines closed subset $C j \subseteq [C j] W_0$.

Furthermore, $\subseteq [C j] \cup [D] \subseteq [C j] \cap [D]$ has the finite intersection property since compact.

$\cap [C j] \cap [D] = \emptyset$. Let $\cap [C j] \cap [D] = \emptyset$.

So we get a semi-norm on $W = W \to V$, $\|\cdot\|$ descends to $V \to H$. 

$V \to \mathbb{H}$.
3. Jech’s Talk (cont.)

3. Hyperimaginaries.

Let $\mathcal{U}$ be a universal domain, let $\alpha < \kappa$ be an ordinal, and $E$ is a type-definable equivalence relation on $\mathcal{U}$.

Let $\mathcal{U}' = \mathcal{U} / E$. Let $\mathcal{X}' = \{(x_{0}, x_{1}, x_{2}, \ldots, y_{0}, y_{1}, y_{2}, \ldots) \mid x_{0} \neq y_{0}, x_{1} \neq y_{1}, \ldots \}$.

Let $\psi(x_{0}, x_{1}, \ldots, y_{0}, y_{1}, y_{2}, \ldots) \in \Delta^{\forall}_{\exists}$.

Interpretation: $\psi_{E}(\bar{a}, \bar{a}, \ldots, \bar{b}, \bar{b}, \bar{b}, \ldots) \iff \exists \bar{b}_{i} \in E(\bar{b}_{i}) \in \mathcal{U}'$.

Lemma: $\mathcal{U}AE$: For $a$ & $b$ fixed,

(i) $tp(\bar{a}E) = tp(\bar{b}E)$

(ii) $\exists c \in E \text{ st. } c \equiv b$

(iii) $\exists c \in E \text{ st. } c \equiv d.$

(i) $\Rightarrow$ (iii) Enough to show that $\forall \psi \in tp(\bar{b}) \iff E(x, a) \rightarrow \psi(x)$.

But $\psi_{E}(x_{E}) \in tp(\bar{b}E) = tp(\bar{a}E)$, $\exists c \in E \text{ st. } \psi(c)$

(ii) $\Rightarrow$ (i) clear.

(iii) $\Rightarrow$ (i) Enough to show for each $\psi_{E} \in tp(\bar{a}E)$, $\psi_{E}(\bar{b}E)$.

By homogeneity, for each $f : c \rightarrow d \in E(f(a), d)$, whence $E(f(a), b)$.
Let \( e \in E(\mathbf{a}, \mathbf{a}) \mathcal{L} \). Then \( E(f(a), f(a)) \leq E(f(c), b) \).

Put \( \psi(f(c)) = f(c) \in E(\mathbf{a}, \mathbf{b}) \mathcal{L} \). \( \square \).

Homogeneity of \( U' \): Start with \( p : \mathbb{A} \to U' \) partial homomorphism.

Same proof as (i) \( \Rightarrow \) (ii) above shows that \( \psi(p(\mathbf{a}) \mathcal{L} p(\mathbf{b})) \leq \psi(p(c_0, c_1, \ldots) \mathcal{L} p(d_0, d_1, \ldots)) \)

\[ \Rightarrow \exists e_0, e_1, \ldots \text{ st. } E(e_0, e_1) \text{ st. } \psi(p(c_0, c_1, \ldots) \mathcal{L} e_0, e_1, \ldots) \leq \psi(p(d_0, d_1, \ldots) \mathcal{L} e_0, e_1, \ldots).

Now use homogeneity in \( \varepsilon U \). i.e. map sending \( a_i \mapsto c_i \) and \( b_i \mapsto e_i \) extends to an automorphism of \( U \).

This extends uniquely to \( U' \).
Compactness of \( W \): From previous lemma, it's enough to check it on sets of predicates.

Suppose \( \exists \phi_i \in \bar{\exists} \) is a set of predicates, finitely realizable in \( W \). Then

\[
\exists = \exists \phi_i(x_0, x_1, \ldots, z^i_0, z^i_1, \ldots) \land \bigwedge y_j 
\]

is finitely realizable in \( W \).

\( \Rightarrow \) realizable in \( W \) by \( x_k = a_k, z^i_j = c^i_j, y_j = b_j \),

where \( \exists \) is realizable in \( W \) by \( a_k, (b_j) \).

\( \Box \)

Back to Hilbert space example...

\( W \) is unit ball in large Hilbert space \( \mathcal{H} \), \( \Delta = \) positive of \( \phi \) formulas on predicates \( \exists \| \leq \lambda \) \iff \( \exists \).

Let \( A \perp B \) mean that \( P_c(A) = P_{cB}(A) \). (\( P_p \) is just projection onto cot a)

Claim: \( A \perp B \) re \( \perp \) is a simple independence relation.

\( \begin{enumerate}
\item \text{Invariance under automorphisms}
\end{enumerate} \)

\[ \text{respects norm, so respects inner products so respects } \perp. \]

Remark \( A \perp B \iff P_c(A) \perp P_c(B) \) where \( L = \mathbb{C} \).

\( A \perp B \iff P_c(A) = P_{cB}(A) \iff P_{cB}(A) \leq L \)

\( \iff P_cP_{cB}(A) = 0 \iff P_{cB}(A) = 0 \iff P_{cB}(P_c(A)) = 0 \)

\( \iff P_L(A) \perp P_L(B) \)
2. Finite character: Use $P_c(A) \perp P_c(B)$ & finiteness.  
   Symmetry: obvious (may need something $\downarrow$. 
   Transitivity: let $L' = \langle B \rangle^\perp$, $A \perp BD \iff P_c(A) \perp P_c(B)$ 
   $\iff P_c(A) \perp P_c(B) \& P_c(A) \perp P_c(D)$ 
   but $P_c(A) \perp P_c(B) \Rightarrow P_c(A) \leq L'$, so $P_c(A) = P_c(A) \perp P_c(D)$, 
   but $P_c(D) = P_c(D) + \text{something in } \langle P_c(B) \rangle$ 
   $\Rightarrow \ P_c(A) \perp P_c(D)$. 

   ie $A \perp BD \iff P_c(A) = P_{BCD}(A) = P_{BC}(A) \iff A \perp B$ 
   $\iff A \perp \frac{C}{BCD}$ 

5. Extension: Given $A, B, C$. Let $L = \langle C \rangle^\perp$. 
   Let $f$ be an automorphism of $A$ fixing $C$ & sending 
   $P_c(A)$ into the orthocomplement of $P_c(B)$ in $L$. 
   Then $A' = f(A)$ has the desired property (since $P_c(A') = P_{c}(f(A)) = f(P_c(B)) \perp P_c(B)$). 

6. Local character: Let $A$ be finite, & $B$ arbitrary. 
   Looking for $B' \subseteq B$ with $|B'| \leq \omega$ so that $A \perp B'$, 
   ie $P_{B'}(A) = P_B(A)$. For each $a \in P_B(A)$, let $b_j$ be a 
   sequence in the finite span of $B$ converging to $A$. 
   Let $B_a = \frac{1}{3}$ all vectors appearing in some $b_j$, $\exists$. Then 
   $\cup B_a = B'$ is what we want. 

7. Independence Thm: 
   Lemma: Every $tp(A/c)$ has a unique orthogonal extn to a 
   type over $CB$. 

Proof: existence we have by extension, so enough to prove uniqueness.

So suppose we have $A \models \exists! x / C$ s.t. $A_1 \vdash B_1$. Then we have a $C$-automorphism sending $A_1$ to $A_1'$ is sending $\mathcal{L}(A)$ into another complement of $\mathcal{L}(B)$ in $C$.

Claim: this determines $\exists! x / C$, because it determines the norm on $<A_1> + <B_1> + <C> = V$.

Suppose we have $v \in V$. Then $\text{dist}(v, C)$ is determined by $\text{tp}(A_1 / C) = \text{tp}(A_1' / C)$.

Let $v = a + b + c$ where $a \in A_1$, $b \in B$, and $c \in C$.

Write $a = a' + \text{PC}(a)$ & $b = b' + \text{PC}(b)$.

Then $||v||^2 = ||a'||^2 + ||b'||^2 + ||a'' + b'' + c||^2$.

We can calculate these by knowing norm $\text{tp}(A_1 / C)$ & $\text{tp}(B / C)$ resp (manually) & $\text{PL}(A) \perp \text{PL}(B)$ (or something like it gives last step).

$\text{PC}(c) = \text{may be } c$. \(\square\)

Proof of ind thm: Assuming $A_1 \equiv A_2$, $B_1 \perp B_2$ & $A_0 \perp B_1$.

$\Rightarrow \exists A_1 \equiv A_2$, $A_0 \perp B_2$.

(Note: this is stronger statement than required for ind thm: in fact end of strong thm)

Let $A \models \exists! x / C$ s.t. $A_1 \perp B_1$. Then from previous lemma,

$A \equiv A_i \models w^5$ (which was what we wanted...) \(\square\)

Note: we didn't need $B_i \perp B_2$.

End of Josh's talk...