Let $\mathcal{M}_1 = (S, \mathcal{I}_1)$, $\mathcal{M}_2 = (S, \mathcal{I}_2)$ be two matroids on common ground set $S$ with rank functions $r_1$ and $r_2$. Many combinatorial optimization problems can be reformulated as the problem of finding the maximum size common independent set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. This problem was studied by Edmonds and Lawler, who proved the following min-max matroid intersection characterization.

**Theorem 1**

\[
\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \subseteq S} (r_1(U) + r_2(S \setminus U)).
\]

As with many min-max characterizations, proving one of the inequalities is straightforward. For any $U \subseteq S$ and $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, we have

\[
|I| \leq |I \cap U| + |I \cap (S \setminus U)| \leq r_1(U) + r_2(S \setminus U),
\]

since $I \cap U$ is an independent set in $\mathcal{I}_1$ and $I \cap (S \setminus U)$ is an independent set in $\mathcal{I}_2$. Therefore,\[
\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| \leq \min_{U \subseteq S} (r_1(U) + r_2(S \setminus U)).
\]

The following important examples illustrate some of the applications of the matroid intersection theorem.

**Examples**

1. For a bipartite graph $G = (V, E)$ with color classes $V = V_1 \cup V_2$, consider $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ where $\mathcal{I}_i = \{F : \forall v \in V_i, \deg_F(v) \leq 1\}$ for $i = 1, 2$. Note that $\mathcal{M}_1$ and $\mathcal{M}_2$ are (partition) matroids, while $\mathcal{I}_1 \cap \mathcal{I}_2$, the set of bipartite matchings of $G$, does not define a matroid on $E$. Also, note that the rank $r_i(F)$ of $F$ in $\mathcal{M}_i$ is the number of vertices in $V_i$ covered by edges in $F$. Then by Theorem 1, the size of a maximum matching in $G$ is

\[
\nu(G) = \min_{U \subseteq E} (r_1(U) + r_2(E \setminus U)) \tag{1}
\]

\[
= \tau(G) \tag{2}
\]

where $\tau(G)$ is the size of a minimum vertex cover of $G$. Thus, the matroid intersection theorem generalizes König’s matching theorem.

2. As a corollary to Theorem 1, we have the following min-max relationship for the minimum common spanning set in two matroids.

\[
\min_{F \text{ spanning in } \mathcal{M}_1 \text{ and } \mathcal{M}_2} |F| = \min_{B_1 \text{ basis in } \mathcal{M}_1} \min_{B_2 \text{ basis in } \mathcal{M}_2} |B_1 \cup B_2|
\]

\[
= \min_{B_1 \text{ basis in } \mathcal{M}_1} |B_1| + |B_2| - |B_1 \cap B_2|
\]

\[
= r_1(S) + r_2(S) - \min_{U \subseteq S} [r_1(U) + r_2(S \setminus U)].
\]

Applying this corollary to the matroids in example 1, it follows that the minimum edge cover in $G$ is equal to the maximum of $|V| - r_1(F) - r_2(E \setminus F)$ over all $F \subseteq E$. Since this is exactly the maximum size of a stable set in $G$, the corollary is a generalization of the König-Rado theorem.

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3. Consider a graph $G$ with a $k$-coloring on the edges, i.e., edge set $E$ is partitioned into color classes $E_1 \cup E_2 \cup \ldots \cup E_k$. The question of whether or not there exists a rainbow spanning tree (i.e. a spanning tree with edges of different colors) can be restated as a matroid intersection problem on $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ with

$$\mathcal{I}_1 = \{F \subseteq E : F \text{ is acyclic}\}$$
$$\mathcal{I}_2 = \{F \subseteq E : |F \cap E_i| \leq 1 \ \forall i\}$$

Since $\mathcal{I}_1 \cap \mathcal{I}_2$ is the set of rainbow forests, there is a rainbow spanning tree of $G$ if and only if

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = |V| - 1.$$

By Theorem 1, this is equivalent to the condition

$$\min_{U \subseteq E} (r_1(U) + r_2(E \setminus U)) = |V| - 1.$$

Since $r_1(U) = |V| - c(U)$ (where $c(U)$ denotes the number of connected components of $(V, U)$), it follows that there is a rainbow spanning tree of $G$ if and only if the number of colors in $E \setminus U$ is at least $c(U) - 1$ for any subset $U \subseteq E$. In other words, a rainbow spanning tree exists if and only if removing the edges of any $t$ colors leaves a graph with at most $t + 1$ components.

4. Given a digraph $G = (V, A)$, a branching $D$ is a subset of arcs such that

(a) $D$ has no directed cycles

(b) For every vertex $v$, $\text{deg}_\text{in}(v) \leq 1$ in $D$.

Branchings are the common independent sets of matroids $\mathcal{M}_1 = (E, \mathcal{I}_1), \mathcal{M}_2 = (E, \mathcal{I}_2)$, where

$$\mathcal{I}_1 = \{F \subseteq E : F \text{ is acyclic in the underlying undirected graph } G\}$$
$$\mathcal{I}_2 = \{F \subseteq E : \text{deg}_\text{in}(v) \leq 1 \ \forall v \in V\}$$

Note that $\mathcal{M}_1$ is a graphic matroid on $G$ and $\mathcal{M}_2$ is a partition matroid. Therefore, the problem of finding a maximum branching of a digraph can be solved by the matroid intersection algorithm.

In order to prove Theorem 1, we need the following lemmas. Recall that a circuit is a minimal dependent set.

**Lemma 2** Let $M = (S, \mathcal{I})$ be a matroid. If $I \in \mathcal{I}, I + x \notin \mathcal{I}$, then $I + x$ contains a unique minimal circuit.

**Lemma 3** (Basis exchange) Suppose $B_1$ and $B_2$ are two bases of a matroid $M$. For any $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that

$$B_1 - x + y \in \mathcal{I} \text{ and } B_2 - y + x \in \mathcal{I}$$

Given an independent set $I$ in a matroid $M = (S, \mathcal{I})$, we define a digraph with vertex set $S$ and arc set $A_M(I) = \{(x, y) : x \in I, y \in S \setminus I, I - x + y \in \mathcal{I}\}$. We often drop the $M$ subscript when referring to $A$. This digraph plays a crucial role in several matroid optimization algorithms including matroid intersection.

**Lemma 4** Let $I, J \in \mathcal{I}$ with $|I| = |J|$. Then $A(I)$ contains a matching on $I \Delta J = (I \setminus J) \cup (J \setminus I)$. 

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Proof: We can assume $I, J$ are bases in $\mathcal{I}$ (otherwise, consider the truncated matroid whose independent sets are those in $\mathcal{I}$ of size less than or equal to $|J|$). We proceed by induction on $|I \setminus J|$. For any $x \in I \setminus J$, there exists $y \in J \setminus I$ such that $J' = J - y + x \in \mathcal{I}$. Then $I \setminus J' = (I \setminus J) - x$ and $J' \setminus I = (J \setminus J) - y$. If $|I \setminus J| = 1$, then we are done; otherwise by induction on $|I \setminus J|$, $A(I)$ contains a matching on $I \Delta J'$, which we extend to a matching of $I \Delta J$ by adding edge $(x, y)$.

Unfortunately, the converse of this theorem is not true, as shown by the following counterexample. Let $\mathcal{M}$ be the graphic matroid on the following graph $G$.

![Graph Diagram]

For $I = \{e_1, e_2, e_3\}, J = \{f_1, f_2, e_3\}$, $A(I)$ contains a matching $(e_1, f_1), (e_2, f_2)$ of $I \Delta J$ and $I \in \mathcal{I}$, but $J \notin \mathcal{I}$.

However, by a slight strengthening of the condition, we can prove the following.

Lemma 5 Given matroid $\mathcal{M} = (S, \mathcal{I}), I \in \mathcal{I}$, and $J \subseteq S$ with $|I| = |J|$, if $A(I)$ contains a unique matching on $I \Delta J$, then $J \in \mathcal{I}$.

Note that in the example above, $A(I)$ also contains the matching $(e_1, f_2), (e_2, f_1)$ on $I \Delta J$, so the stronger condition fails.

Proof: Let $N$ denote the unique perfect matching on $I \Delta J$ and consider the digraph in which we reverse the orientation of the arcs in $N$. By the uniqueness of the perfect matching, there are no directed cycles in the resulting graph, so there is a topological ordering of the vertices. This ordering induces a labeling on vertices in $N = \{(y_1, z_1), (y_2, z_2), \ldots (y_t, z_t)\}$ such that there are no arcs $(y_i, z_j)$ for $i < j$.

If $J \notin \mathcal{I}$, then it contains a circuit $C$. Let $i$ be the smallest index such that $z_i \in C$. Since there are no arcs from $y_i$ to $z_j$ with $j > i$, $I - y_i + z_j \notin \mathcal{I}$, implying $z_j \in \text{span}(I - y_i)$. Since this is true for all $j > i$, $C - z_i \subseteq \text{span}(I - y_i)$. But since $C$ is a circuit, $z_i \in \text{span}(C - z_i) \subseteq \text{span}(I - y_i)$. Then $I - y_i + z_i \notin \mathcal{I}$ and by definition of $A(I)$, $(y_i, z_i) \notin A(I)$ (since $I - y_i + z_i \notin \mathcal{I}$), a contradiction to the existence of perfect matching $N$. Therefore $J \in \mathcal{I}$.

Now, we state the matroid intersection algorithm, whose proof we will give in the next lecture. Since $\mathcal{I}$ may be exponential in size, we assume our matroid is described by an oracle which, given $I \subseteq S$, can determine in polynomial time if $I \in \mathcal{I}$. Then the running time of the algorithm is polynomial in the number of calls to the oracle.

First, for $I \subseteq S$, define the digraph $D(I) = (S, A)$ as follows: for $y \in I, x \notin I$, we have an arc $(y, x) \in A$ if $I - y + x \in \mathcal{I}_1$ and $(x, y) \in A$ if $I - y + x \in \mathcal{I}_2$. This is the union of the arcset $A_{M_1}(I)$ corresponding to $\mathcal{I}_1$ and the reverse of the arcset $A_{M_2}(I)$ corresponding to $\mathcal{I}_2$. Consider the sets

$$X_1 = \{x \in S \setminus I : I + x \in \mathcal{I}_1\}, X_2 = \{x \in S \setminus I : I + x \in \mathcal{I}_2\}.$$

**Matroid Intersection Algorithm**

- **Input** Matroids $\mathcal{M}_1 = (S, \mathcal{I}_1), \mathcal{M}_2 = (S, \mathcal{I}_2)$
- **Output** $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ of maximum size

$\begin{align*}
I &\leftarrow \emptyset \\
\text{while } &D(I) \text{ has a path from } X_1 \text{ to } X_2 \\
&\quad I \leftarrow I \Delta \mathcal{V}(P), \text{ where } P \text{ is a shortest path from } X_1 \text{ to } X_2
\end{align*}$

We will prove the correctness of this algorithm in the next lecture.