18.997 Topics in Combinatorial Optimization April 8, 2004

Lecture 16

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This lecture is about jump systems. While they are briefly discussed in chapter 41 of Schrijver's book, they are not covered extensively. A good reference for this material is a set of notes by Jim Geelen, available at `http://www.math.uwaterloo.ca/~jfgeelen/publications/js.ps`.

1 The Basics of Jump Systems

We begin with some notational definitions that will simplify the rest of our discussion.

Definition 1. For $x, y \in \mathbb{Z}^n$, we let the *box* $[x, y]$ be the set

$$\{z \in \mathbb{Z}^n | \min(x_i, y_i) \leq z_i \leq \max(x_i, y_i) \forall i \in \{1, \dots, n\}\}.$$

In that which follows, we shall use the L_1 metric unless otherwise stated. That is, for $x, y \in \mathbb{Z}^n$, we shall have

$$d(x, y) = ||x - y||_1 = \sum_{i=1}^{n} |x_i - y_i|.$$

Definition 2. For $x, y \in \mathbb{Z}^n$, a *step x' from x to y* is a point $x' \in \mathbb{Z}^n$ such that $x' \in [x, y]$ and $d(x, x') = 1$.

We can now define our main objects of study:

Definition 3. A *jump system* is a set $J \subseteq \mathbb{Z}^n$ such that if x' is a step from x to y, then either

1. $x' \in J$, or
2. There exists a step x'' from x' to y such that $x'' \in J$.

We now consider some examples of jump systems.

Example 1. Let M be a matroid over a set S, $|S| = n$. We let each coordinate of $\mathbb{Z}^n$ correspond to an element of S, and let $J \subseteq \{0, 1\}^n$ be the set of characteristic functions for bases of M,

$$J = \{\chi^B | B \text{ a basis of } M\}.$$

Claim 1. *The set J is a jump system.*

Proof. Let x and y be the respective characteristic vectors of two bases b_1 and b_2 of M. A step x' from x to y corresponds to either:

1. Adding to b_1 an element of $b_2 \setminus b_1$, or
2. Removing from b_1 some element of $b_1 \setminus b_2$.

Since all bases have the same size, the set corresponding to x' will never be a basis, so $x' \notin J$. We thus require there to be some step x'' from x' to y such that $x'' \in J$, i.e., such that the set corresponding to x'' is a basis. In both cases, this is guaranteed by Basis Exchange (see lecture 11). □

Example 2. Let $G = (V, E)$ be an undirected graph, and let $n = |V|$. For every subgraph $H = (V, F)$ of G, we can construct its *degree sequence* $d_H \in \mathbb{Z}^n$ by setting the i^{th} coordinate of d_H equal to the degree in H of the i^{th} vertex of G. Now let

$$J = \{d_H | H \text{ is a subgraph of } G\}$$

be the set of all degree sequences of subgraphs of G.

Claim 2. *J is a jump system.*

We could check this directly, but it would be rather tedious. Instead, we shall describe several operations that one can perform on jump systems that give rise to other jump systems. We will then show how to construct J using these operations, from which Claim 2 will follow.

2 Operations on Jump Systems

In this section, we describe several operations on jump systems. Throughout the sequel, let $J \subseteq \mathbb{Z}^n$ be a jump system.

Translation If J is a jump system, $J + a$ is a jump system for any vector $a \in \mathbb{Z}^n$.

Reflection For some $i \in \{1, \dots, n\}$ reflect the entire jump system through the $x_i = 0$ plane, replacing each point $(x_1, \dots, x_n) \in J$ with $(x_1, \dots, -x_i, \dots, x_n)$. This clearly produces another jump system.

Projection Project onto some axis-parallel subspace of $\mathbb{Z}^n$. That is, let $S \subseteq \{1, \dots, n\}$ be some subset of the coordinates, and create a new set $J_S = \{x|_S, x \in J\}$. To see that J_S is a jump system, note that the projection of a step either gives a step or no motion at all.

Sum If J_1 and J_2 are jump systems, define a new set $J_1 + J_2 := \{x + y | x \in J_1, y \in J_2\}$.

Claim 3. *$J_1 + J_2$ is a jump system.*

Proof. Let $x_1, y_1 \in J_1$, $x_2, y_2 \in J_2$, $x = x_1 + x_2$, $y = y_1 + y_2$, and suppose that x' is a step from x to y. We shall show that either $x' \in J_1 + J_2$ or that there exists a step $x'' \in J$ from x' to y.

Take $z_1 \in J_1$, $z_2 \in J_2$ such that $d(x', z_1 + z_2) = 1$ and so that $d(z_1, y_1) + d(z_2, y_2)$ is minimized. This is always possible, since $d(x', x_1 + x_2) = 1$. We now have two possibilities:

Case 1: $z_1 + z_2 \in [x', y]$.

In this case, we are already done, since $z_1 + z_2 \in J$, and $z_1 + z_2$ is a step from x' to y.

Case 2: $x' \in [z_1 + z_2, y]$.

Let $x' = z_1 + z_2 + s$, so that $z_1 + z_2 + s \in [z_1 + z_2, y_1 + y_2]$. This implies that either $z_1 + s \in [z_1, y_1]$ or $z_2 + s \in [z_2, y_2]$. By symmetry, we may assume without loss of generailty that $z_1 + s \in [z_1, y_1]$. We now have two possibilities:

1. $z_1 + s \in J_1$, or
2. $z_1 + s \notin J_1$.

In the first case, we are done, since $x' = (z_1 + s) + z_2 \in J_1 + J_2$, as required. In the second case, the fact that J_1 is a jump system implies that there exists a step $z_1' \in J_1$ from $z_1 + s$ to y. It thus follows easily that $d(z_1' + y_1) + d(z_2, y_2) < d(z_1, y_1) + d(z_2, y_2)$. However, since $d(z_1' + z_2, x') = 1$, and since $z_1 + z_2$ was chosen from points satisfying this condition so as to minimize $d(z_1, y_1) + d(z_2, y_2)$, we must have $d(z_1', y_1) + d(z_2, y_2) \geq d(z_1, y_1) + d(z_2, y_2)$, thereby resulting in a contradiction.

□

We can now prove Claim 2 and show that the set described in Example 2 is in fact a jump system.

Proof of Claim 2. If G is a single edge connecting two vertices i and j, the degree sequences are just $\{0, e_i + e_j\}$, which is obviously a jump system. If G is a more complicated graph, its set of degree sequences is just the sum of the jump systems for each of its edges, which is a jump sequence by Claim 3. □

3 Optimizing Over a Jump System

Suppose we have some vector (w_i), $i = 1, \ldots, n$. In this section, we shall show how to maximize $w^T x$ for $x \in J$. By reflecting and reordering the coordinates if necessary, we may assume $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$. We shall find the optimum with the following greedy algorithm:

```
J_0 = J
For i=1 to n
  J_i ← arg max_{x∈J_{i-1}} x_i
Return J_n,
```

where the argmax returns the set of all values achieving the maximum.

Claim 4. *This algorithm returns the desired maximum.*

Proof. Suppose to the contrary. By induction on i, we can assume that the maximum taken over J is not equal to that taken over J_1. Call the vector that achieves the former quantity x, and call the vector that achieves the latter y. Our assumption implies that $x_1 < y_1$.

We have also assumed that $w^T x > w^T y$. Now, either

1. $x + e_1 \in J$, or
2. $x + e_1 \pm e_k \in J$.

Take x out of the optimal solutions to be the one that minimizes $y_1 - x_1$. In the first case, we have

$$w^T(x + e_1) = w^T x + w_1 \geq w^T x,$$

which yields a contradiction. In the second case, we have

$$w^T(x + e_1 \pm e_k) = w^T x + w_1 \pm w_k \geq w^T x,$$

which again yields a contradiction. This completes the proof. □

Observe that since we first computed the absolute values of the w_i's, the greedy algorithm described here does not correspond to the classical greedy algorithm to compute a maximum weight basis of a matroid.

4 Membership in Jump Systems

The material in this section is almost exclusively due to Lovász. A paper covering this and much more is available at `http://research.microsoft.com/users/lovasz/jump.ps`.

Suppose we are given a jump system J in some sort of implicit form. In this section, we take up the question of when we can determine whether some point x is in J. This specializes to many standard problems that we have already considered in this class.

Example 3. Let G be a graph, and let J_G be its set of degree sequences. Asking whether the vector $(1, \ldots, 1) \in J_G$ is equivalent to asking whether there is some subgraph of G in which every vertex of G has degree exactly one. This is exactly the question of whether G admits a perfect matching.

This can be generalized to the "factor problem": Given a graph $G = (V, E)$ and a function $f : V \to \mathbb{Z}_{\geq 0}$, does there exist a subgraph $F \subseteq E$ such that $d_f(v) = f(v)$ for all v? (It turns out that, using classical methods, one can reduce this problem to solving perfect matching, so this doesn't really gain us too much generality.)

Example 4. Let M_1 and M_2 be matroids, and let J_1 and J_2 be their respective jump systems of bases, as described in Example 1. Now let $J = J_1 - J_2$. The question of whether 0 belongs to J is equivalent to asking whether M_1 and M_2 have a common basis.

So it would be great if we could get a good general solution to the membership problem for jump systems. Unfortunately, this is going to turn out to be too much to ask for. As the next example will show, the membership problem includes matroid matching, which we established in an earlier lecture to be NP-hard.

Example 5. Let $M = (S, \mathcal{I})$ be a matroid, let E be a set of pairs of elements of S, let J_M be the jump system of bases of M (as in Example 1), and let J_G be the jump system of degree sequences of $G = (S, E)$ (as in Example 2). Now, let $J = J_M - J_G$. Every vector in J_M is a $\{0, 1\}$-vector of weight equal to the rank of M. Such a vector, when interpreted as an element of J_G, corresponds to a matching of weight $\text{rk}(M)$. It thus follows that $0 \in J$ if and only if there is a matching in G of weight $\text{rk}(M)$ that is independent in M. Checking if $0 \in J$ is thus precisely equivalent to matroid matching, which, alas, is NP-hard.

We therefore can't hope to solve the membership problem in general. However, Lovász described a broad class of cases where we can solve it, which we will discuss here and in the next lecture.

4.1 The Beginnings of a Min-Max Relation

It will be useful to generalize a little bit and consider the question of finding the closest element of a jump system to some box. Let J be a jump system, and let $B = [a, b]$, $a, b \in \mathbb{Z}^n$, be a box. Now consider the quantity

$$d(J, B) = \min_{x \in J, y \in B} d(x, y) = \min_{x \in J, y \in B} \sum_i |x_i - y_i|.$$

If $w \in \{0, +1, -1\}^n$, then clearly

$$d(J, B) \geq \min_{x \in J, y \in B} w^T(x - y) = \min_{x \in J} w^T x - \max_{y \in B} w^T y. \tag{1}$$

In some classes of systems, we will have equality in Equation (1), which will facilitate a solution to the membership problem. To state when this occurs, we will need some terminology. First, given a box B, let

$$J_B = \{x \in J | d(x, B) = d(J, B)\}.$$

Theorem 1 (Lovász). *J_B is a jump system.*

Now, we will define two sets, V_B^+ and V_B^-:

$$V_B^+ = \{i \in \{1, \ldots, n\} \mid \exists x \in J_B \text{ s.t. } x_i > b_i\},$$

$$V_B^- = \{i \in \{1, \ldots, n\} \mid \exists x \in J_B \text{ s.t. } x_i < a_i\}.$$

The main theorem, which we shall show in the next lecture, is:

Theorem 2. *If $V_B^+ \cap V_B^- = \emptyset$, then we have equality in Equation (1).*