PROBLEMS ON RECURRENCES

1. Let $T_0 = 2, T_1 = 3, T_2 = 6$, and for $n \geq 3$,
   \[ T_n = (n + 4)T_{n-1} - 4nT_{n-2} + (4n - 8)T_{n-3}. \]

   The first few terms are: 2, 3, 6, 14, 40, 152, 784, 40576, 363392. Find, with proof, a formula for $T_n$ of the form $T_n = A_n + B_n$, where \{A_n\} and \{B_n\} are well-known sequences.

2. For which real numbers $a$ does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Express the answer in simplest form.)

3. Prove or disprove that there exists a positive real number $u$ such that $\lfloor u^n \rfloor - n$ is an even integer for all positive integers $n$. (Here, $\lfloor x \rfloor$ is the greatest integer $\leq x$.)

4. Define $u_n$ by $u_0 = 0, u_1 = 4$, and $u_{n+2} = \frac{6}{5}u_{n+1} - u_n$. Show that $|u_n| \leq 5$ for all $n$. (In fact, $|u_n| < 5$ for all $n$. Can you show this?)

5. Show that the next integer above $(\sqrt{3} + 1)^{2n}$ is divisible by $2^{n+1}$.

6. Let $a_0 = 0, a_1 = 1$, and for $n \geq 2$ let $a_n = 17a_{n-1} - 70a_{n-2}$. For $n \geq 6$, show that the first (most significant) digit of $a_n$ (when written in base 10) is a 3.

7. Let $a, b, c$ denote the (real) roots of the polynomial $P(t) = t^3 - 3t^2 - t + 1$. If $u_n = a^n + b^n + c^n$, what linear recursion is satisfied by \{u_n\}? If $a$ is the largest of the three roots, what is the closest integer to $a^5$?

8. Solve the first order recursion given by $x_0 = 1$ and $x_n = 1 + (1/x_{n-1})$. Does \{x_n\} approach a limiting value as $n$ increases?

9. If $u_0 = 0, u_1 = 1$, and $u_{n+2} = 4(u_{n+1} - u_n)$, find $u_{16}$.

10. Let $a_0 = 1, a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of $a_{2015}$.

11. Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute
    \[ \prod_{i=0}^{\infty} \left( 1 - \frac{1}{a_k} \right) \]
    in closed form.

12. (a) Define $u_0 = 1, u_1 = 1$, and for $n \geq 1$,
    \[ 2u_{n+1} = \sum_{k=0}^{n} \binom{n}{k} u_k u_{n-k}. \]
Find a simple expression for \( F(x) = \sum_{n \geq 0} u_n \frac{x^n}{n!} \). Express your answer in the form \( G(x) + H(x) \), where \( G(x) \) is even (i.e., \( G(-x) = G(x) \)) and \( H(x) \) is odd (i.e., \( H(-x) = -H(x) \)).

(b) Define \( u_0 = 1 \) and for \( n \geq 0 \),

\[
2u_{n+1} = \sum_{k=0}^{n} \binom{n}{k} u_k u_{n-k}.
\]

Find a simple expression for \( u_n \).

13. For a positive integer \( n \) and any real number \( c \), define \( x_k \) recursively by \( x_0 = 0, x_1 = 1 \), and for \( k \geq 0 \),

\[
x_{k+2} = \frac{cx_{k+1} - (n - k)x_k}{k + 1}.
\]

Fix \( n \) and then take \( c \) to be the largest value for which \( x_{n+1} = 0 \). Find \( x_k \) in terms of \( n \) and \( k \), \( 1 \leq k \leq n \).

14. Let \( f(x) \) be a polynomial with integer coefficients. Define a sequence \( a_0, a_1, \ldots \) of integers such that \( a_0 = 0 \) and \( a_{n+1} = f(a_n) \) for all \( n \geq 0 \). Prove that if there exists a positive integer \( m \) for which \( a_m = 0 \) then either \( a_1 = 0 \) or \( a_2 = 0 \).

15. Define a sequence by \( a_0 = 1 \), together with the rules \( a_{2n+1} = a_n \) and \( a_{2n+2} = a_n + a_{n+1} \) for each integer \( n \geq 0 \). Prove that every positive rational number appears in the set

\[
\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \ldots \right\}.
\]

16. Let \( 1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots \) be a sequence defined by \( x_k = k \) for \( k = 1, 2, \ldots, 2006 \) and \( x_{k+1} = x_k + x_{k-2005} \) for \( k \geq 2006 \). Show that the sequence has 2005 consecutive terms each divisible by 2006.

17. Let \( a_1 < a_2 \) be two given integers. For any integer \( n \geq 3 \), let \( a_n \) be the smallest integer which is larger than \( a_{n-1} \) and can be uniquely represented as \( a_i + a_j \), where \( 1 \leq i < j \leq n - 1 \). Given that there are only a finite number of even numbers in \( \{a_n\} \), prove that the sequence \( \{a_{n+1} - a_n\} \) is eventually periodic, i.e. that there exist positive integers \( T, N \) such that for all integers \( n > N \), we have

\[
a_{T+n+1} - a_{T+n} = a_{n+1} - a_n.
\]

18. Let \( k \) be an integer greater than 1. Suppose that \( a_0 > 0 \), and define

\[
a_{n+1} = a_n + \frac{1}{\sqrt[n]{a_n}}
\]

for \( n > 0 \). Evaluate

\[
\lim_{n \to \infty} \frac{a_{n+1}^k}{n^k}.
\]
19. Let $x_0 = 1$ and for $n \geq 0$, let $x_{n+1} = 3x_n + \lfloor x_n \sqrt{5} \rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for $x_{2007}$. ([$a$] means the largest integer $\leq a$.)

20. (a) Let $a_0, \ldots, a_{k-1}$ be real numbers, and define

$$a_n = \frac{1}{k} (a_{n-1} + a_{n-2} + \cdots + a_{n-k}), \ n \geq k.$$  

Find $\lim_{n \to \infty} a_n$ (in terms of $a_0, a_1, \ldots, a_{k-1}$).

(b) Somewhat more generally, let $u_1, \ldots, u_k \geq 0$ with $\sum u_i = 1$ and $u_k \neq 0$. Assume that the polynomial $x^k - u_1 x^{k-1} - u_2 x^{k-2} - \cdots - u_k$ cannot be written in the form $P(x^d)$ for some polynomial $P$ and some $d > 1$. Now define

$$a_n = u_1 a_{n-1} + u_2 a_{n-2} + \cdots + u_k a_{n-k}, \ n \geq k.$$  

Again find $\lim_{n \to \infty} a_n$. (Part (a) is the case $u_1 = \cdots = u_k = 1/k$.)

21. (a) (repeats Congruence and Divisibility Problem #22) Define $u_n$ recursively by $u_0 = u_1 = u_2 = u_3 = 1$ and

$$u_n u_{n-4} = u_{n-1} u_{n-3} + u_{n-2}^2, \ n \geq 4.$$  

Show that $u_n$ is an integer.

(b) Do the same for $u_0 = u_1 = u_2 = u_3 = u_4 = 1$ and

$$u_n u_{n-5} = u_{n-1} u_{n-4} + u_{n-2} u_{n-3}, \ n \geq 5.$$  

(c) (much harder) Do the same for $u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = 1$ and

$$u_n u_{n-6} = u_{n-1} u_{n-5} + u_{n-2} u_{n-4} + u_{n-3}^2, \ n \geq 6,$$

and for $u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 1$ and

$$u_n u_{n-7} = u_{n-1} u_{n-6} + u_{n-2} u_{n-5} + u_{n-3} u_{n-4}, \ n \geq 7.$$  

(d) What about $u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = u_7 = 1$ and

$$u_n u_{n-8} = u_{n-1} u_{n-7} + u_{n-2} u_{n-6} + u_{n-3} u_{n-5} + u_{n-4}^2, \ n \geq 8?$$  

22. (very difficult) Let $a_0, a_1, \ldots$ satisfy a homogeneous linear recurrence (of finite degree) with constant coefficients. I.e., for some complex (or real, if you prefer) numbers $\nu_1, \ldots, \nu_k$ we have

$$a_n = \nu_1 a_{n-1} + \cdots + \nu_k a_{n-k}$$  

for all $n \geq k$. Define

$$b_n = \begin{cases} 1, & a_n \neq 0 \\ 0, & a_n = 0. \end{cases}$$  

Show that $b_n$ is eventually periodic, i.e., there exists $p > 0$ such that $b_n = b_{n+p}$ for all $n$ sufficiently large.