18.S34 (FALL 2007)
LIMIT PROBLEMS

1. Let $a$ and $b$ be positive real numbers. Prove that

$$\lim_{n \to \infty} (a^n + b^n)^{1/n}$$

equals the larger of $a$ and $b$. What happens when $a = b$?

2. Show that $\lim_{n \to \infty} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n))$ exists and lies between $\frac{1}{2}$ and 1.

**NOTE.** This number, known as *Euler’s constant* and denoted $\gamma$, is probably the third most important constant in the theory of complex variables, after $\pi$ and $e$. Numerically we have

$$\gamma = 0.577215664901532860651209008240243104215933593992\cdots.$$ 

It is a famous unsolved problem to decide whether $\gamma$ is irrational.

3. (47P) If $(a_n)$ is a sequence of numbers such that, for $n \geq 1$,

$$(2 - a_n)a_{n+1} = 1,$$

prove that $\lim_{n \to \infty} a_n$ exists and equals 1.

4. Let $K$ be a positive real number. Take an arbitrary positive real number $x_0$ and form the sequence

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{K}{x_n} \right).$$

Show that $\lim_{n \to \infty} x_n = \sqrt{K}$. (**REMARK.** this is how most calculators determine $\sqrt{K}$.)

5. (70P) Given a sequence $(x_n)$ such that $\lim_{n \to \infty} (x_n - x_{n-2}) = 0$, prove that

$$\lim_{n \to \infty} \frac{x_n - x_{n-1}}{n} = 0.$$

6. Let $x_{n+1} = x_n^2 - 6x_n + 10$. For what values of $x_0$ is \{x_n\} convergent, and how does the value of the limit depend on $x_0$?
7. (90P) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt{n} - \sqrt{m}$, $(n, m = 0, 1, 2, \ldots)$? Justify your answer.

8. Let $x_0 = 1$ and $x_{n+1} = x_n + 10^{-10^n}$. Does $\lim_{n\to\infty} x_n$ exist? Explain.

9. (00P) Show that the improper integral

$$\lim_{B \to \infty} \int_0^B \sin(x) \sin(x^2) \, dx$$

converges.

10. Let $x > 0$. Define $a_1 = x$ and $a_{n+1} = x^{a_n}$ for $n \geq 1$. For which $x$ does $\lim_{n\to\infty} a_n$ exist (and is finite)?

**PART II**

**LIMITS.** Two useful techniques are:

(a) *L'Hôpital’s rule.* If $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0$, then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)},$$

provided the derivatives in question exist. Some limits can be converted to this form by first taking logarithms, or by substituting $1/x$ for $x$, etc.

(b) If $f(x)$ is reasonably well-behaved (e.g., continuous) on the closed interval $[a, b]$, then

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) = \int_a^b f(x) \, dx,$$

where the limit is over any sequence of “partitions of $[a, b]$” $a = x_0 < x_1 < \cdots < x_n = b$ such that the maximum value of $x_i - x_{i-1}$ approaches 0. In particular, taking $a = 0$, $b = 1$, $x_i = i/n$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(i/n) = \int_0^1 f(x) \, dx.$$

Sometimes a limit of products can be converted to this form by taking logarithms.

The next problems are all from the Putnam Exam.
11. Let $a > 0$, $a \neq 1$. Find
\[
\lim_{x \to \infty} \left( \frac{1}{x} \frac{a^x - 1}{a - 1} \right)^{1/x}
\]

12. Find
\[
\lim_{n \to \infty} \left[ \frac{1}{n^4} \prod_{i=1}^{2n} (n^2 + i^2)^{1/n} \right]
\]

13. Let $0 < a < b$. Evaluate
\[
\lim_{t \to 0} \left[ \int_0^1 (bx + a(1 - x))^t \, dx \right]^{1/t}
\]

14. Evaluate
\[
\lim_{x \to 0} \frac{1}{x} \int_0^x (1 + \sin(2t))^{1/t} \, dt
\]

15. Evaluate
\[
\lim_{n \to \infty} \sum_{j=1}^{n^2} \frac{n}{n^2 + j^2}
\]

16. Evaluate
\[
\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}
\]

17. Evaluate
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left( \left\lfloor \frac{2n}{k} \right\rfloor - 2 \left\lfloor \frac{n}{k} \right\rfloor \right)
\]

Express your answer in the form $\log(a) - b$, where $a$ and $b$ are positive integers.

18. Evaluate
\[
\sqrt{2207 - \frac{1}{2207 - \frac{1}{2207 - \ldots}}}
\]

Express your answer in the form $\frac{a + b\sqrt{c}}{d}$, where $a, b, c, d$ are integers.
19. Assume that \((a_n)_{n\geq 1}\) is an increasing sequence of positive real numbers such that \(\lim a_n/n = 0\). Must there exist infinitely many positive integers \(n\) such that \(a_{n-i} + a_{n+i} < 2a_n\) for \(i = 1, 2, \ldots, n-1\)?

20. Evaluate
\[
\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left( \frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.
\]

21. Let \(k\) be an integer greater than 1. Suppose \(a_0 > 0\), and define
\[
a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}
\]
for \(n > 0\). Evaluate
\[
\lim_{n \to \infty} \frac{a_{n+1}^{k+1}}{n^k}.
\]