10.3.2 The semidefinite relaxation

We will now present a semidefinite relaxation for (108) (see [BCSZ14]).

Let us identify $R_l$ with the $L \times L$ permutation matrix that cyclicly permutes the entries of a vector by $l$ coordinates:

$$R_l \begin{bmatrix} u_1 \\ \vdots \\ u_L \end{bmatrix} = \begin{bmatrix} u_{1-l} \\ \vdots \\ u_{L-l} \end{bmatrix}.$$

This corresponds to an $L$-dimensional representation of the cyclic group. Then, (108) can be rewritten:

$$\sum_{i,j \in [n]} \langle R_{-l_i}y_i, R_{-l_j}y_j \rangle = \sum_{i,j \in [n]} (R_{-l_i}y_i)^T R_{-l_j}y_j
= \sum_{i,j \in [n]} \text{Tr} \left[(R_{-l_i}y_i)^T R_{-l_j}y_j\right]
= \sum_{i,j \in [n]} \text{Tr} \left[y_i^T R_{-l_i}^T R_{-l_j}y_j\right]
= \sum_{i,j \in [n]} \text{Tr} \left[(y_iy_j^T)^T R_{l_i}R_{l_j}^T\right].$$

We take

$$X = \begin{bmatrix} R_{l_1} \\ R_{l_2} \\ \vdots \\ R_{l_n} \end{bmatrix} \begin{bmatrix} R_{l_1}^T & R_{l_2}^T & \cdots & R_{l_n}^T \end{bmatrix} \in \mathbb{R}^{nL \times nL}, \tag{109}$$

and can rewrite (108) as

$$\text{max} \quad \text{Tr}(CX)$$
$$\text{s. t.} \quad X_{ii} = I_{L \times L}$$
$$X_{ij} \text{ is a circulant permutation matrix}$$
$$X \succeq 0$$
$$\text{rank}(X) \leq L, \tag{110}$$

where $C$ is the rank 1 matrix given by

$$C = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \begin{bmatrix} y_1^T & y_2^T & \cdots & y_n^T \end{bmatrix} \in \mathbb{R}^{nL \times nL}, \tag{111}$$

with blocks $C_{ij} = y_i y_j^T$. 

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The constraints $X_{ii} = I_{L \times L}$ and $\text{rank}(X) \leq L$ imply that $\text{rank}(X) = L$ and $X_{ij} \in O(L)$. Since the only doubly stochastic matrices in $O(L)$ are permutations, (110) can be rewritten as

$$\begin{align*}
\text{max} & \quad \text{Tr}(CX) \\
\text{s. t.} & \quad X_{ii} = I_{L \times L} \\
& \quad X_{ij} \mathbf{1} = \mathbf{1} \\
& \quad X_{ij} \text{ is circulant} \\
& \quad X \geq 0 \\
& \quad X \succeq 0 \\
& \quad \text{rank}(X) \leq L.
\end{align*}$$

(112)

Removing the nonconvex rank constraint yields a semidefinite program, corresponding to (??),

$$\begin{align*}
\text{max} & \quad \text{Tr}(CX) \\
\text{s. t.} & \quad X_{ii} = I_{L \times L} \\
& \quad X_{ij} \mathbf{1} = \mathbf{1} \\
& \quad X_{ij} \text{ is circulant} \\
& \quad X \geq 0 \\
& \quad X \succeq 0.
\end{align*}$$

(113)

Numerical simulations (see [BCSZ14, BKS14]) suggest that, below a certain noise level, the semidefinite program (113) is tight with high probability. However, an explanation of this phenomenon remains an open problem [BKS14].

**Open Problem 10.3** For which values of noise do we expect that, with high probability, the semidefinite program (113) is tight? In particular, is it true that for any $\sigma$ by taking arbitrarily large $n$ the SDP is tight with high probability?

**References**

