4.7.2 $k$-lifts of graphs

Given a graph $G$, on $n$ nodes and with max-degree $\Delta$, and an integer $k \geq 2$ a random $k$ lift $G^\otimes k$ of $G$ is a graph on $kn$ nodes obtained by replacing each edge of $G$ by a random $k \times k$ bipartite matching. More precisely, the adjacency matrix $A^\otimes k$ of $G^\otimes k$ is a $nk \times nk$ matrix with $k \times k$ blocks given by

$$A^\otimes k_{ij} = A_{ij} \Pi_{ij},$$

where $\Pi_{ij}$ is uniformly randomly drawn from the set of permutations on $k$ elements, and all the edges are independent, except for the fact that $\Pi_{ij} = \Pi_{ji}$. In other words,

$$A^\otimes k = \sum_{i<j} A_{ij} \left( e_i e_j^T \otimes \Pi_{ij} + e_j e_i^T \otimes \Pi_{ij}^T \right),$$

where $\otimes$ corresponds to the Kronecker product. Note that

$$\mathbb{E} A^\otimes k = A \otimes \left( \frac{1}{k} J \right),$$

where $J = \mathbf{1} \mathbf{1}^T$ is the all-ones matrix.

Open Problem 4.5 (Random $k$-lifts of graphs) *Give a tight upperbound to*

$$\mathbb{E} \left\| A^\otimes k - \mathbb{E} A^\otimes k \right\|.$$

Oliveira [Oli10] gives a bound that is essentially of the form $\sqrt{\Delta \log (nk)}$, while the results in [ABG12] suggest that one may expect more concentration for large $k$. It is worth noting that the case of $k = 2$ can essentially be reduced to a problem where the entries of the random matrix are independent and the results in [BvH15] can be applied to, in some case, remove the logarithmic factor.

References


