18.S34 PROBLEMS #12

Fall 2007

127. [1] Find all solutions in integers to (a) $x + y = xy$, (b) $x + y + 1 = xy$, (c) $x^2 + y^2 = xy + x + y$.

128. [1.5] Choose 23 people at random. What is the probability some two of them have the same birthday? (You may ignore the existence of February 29.)

129. [1] Let $p$ and $q$ be consecutive odd primes (i.e., no prime numbers are between them). Show that $p + q$ is a product of at least three primes. For instance, $23 + 29$ is the product of the three primes 2, 2, and 13.

130. [3.5] Evaluate in closed form:

$$\int \frac{x \, dx}{\sqrt{x^4 + 4x^3 - 6x^2 + 4x + 1}}.$$

131. [1.5] Suppose that for each $n \geq 1$, $f_n(x)$ is a continuous function on the closed interval $[0, 1]$. Suppose also that for any $x \in [0, 1]$,

$$\lim_{n \to \infty} f_n(x) = 0.$$

Is it then true that

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0?$$

132. Let $x \geq 1$ be a real number. Let $f(x)$ be the maximum number of $1 \times 1$ squares that can fit inside an $x \times x$ square without overlap. (It is not assumed that the sides of the $1 \times 1$ squares are parallel to the sides of the $x \times x$ square.) For instance, if $x$ is an integer then $f(x) = x^2$.

(a) [3] Show that for some values of $x$, $f(x) > \lfloor x \rfloor^2$, where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

(b) [5] Find a formula for (or at least a method of computing) $f(x)$ for any $x$.

133. [3] Let $S$ be any finite set of points in the plane such that not all of them lie on a single straight line. Show that some (infinite) line intersects exactly two points of $S$.  

1
134. [2.5] (The non-messing-up-theorem) Let $M$ be an $m \times n$ matrix of integers. For example,

$$M = \begin{bmatrix}
7 & 3 & 1 & 4 & 2 \\
5 & 6 & 3 & 1 & 5 \\
2 & 2 & 1 & 8 & 4
\end{bmatrix}.$$

Rearrange the rows of $M$ in increasing order.

$$\begin{bmatrix}
1 & 2 & 3 & 4 & 7 \\
1 & 3 & 5 & 5 & 6 \\
1 & 2 & 2 & 4 & 8
\end{bmatrix}.$$

Now rearrange the columns in increasing order.

$$\begin{bmatrix}
1 & 2 & 2 & 4 & 6 \\
1 & 2 & 3 & 4 & 7 \\
1 & 3 & 5 & 5 & 8
\end{bmatrix}.$$

Show that the rows remain in increasing order.

135. (a) [2.5] Let $a(n)$ be the number of ways to write the positive integer $n$ as a sum of distinct positive integers, where the order of the summands is not taken into account. Similarly let $b(n)$ be the number of ways to write $n$ as a sum of odd positive integers, without regard to order. For instance, $a(7) = 5$, since $7 = 6 + 1 = 5 + 2 = 4 + 3 = 4 + 2 + 1$; while $b(n) = 5$, since $1 + 1 + 1 + 1 + 1 + 1 + 1 = 3 + 1 + 1 + 1 + 1 = 3 + 3 + 1 = 5 + 1 + 1 = 7$. Show that $a(n) = b(n)$ for all $n$.

(b) [3] Let $A$ and $B$ be subsets of the positive integers. Let $a_A(n)$ be the number of ways to write $n$ as a sum (without regard to order) of distinct elements of the set $A$. Let $b_B(n)$ be the number of ways to write $n$ as a sum (without regard to order) of elements of $B$. Call $(A, B)$ an Euler pair if $a_A(n) = b_B(n)$ for all $n$. For instance, (a) above states that if $A$ consists of all positive integers and $B$ consists of the odd positive integers, then $(A, B)$ is an Euler pair. Show that $(A, B)$ is an Euler pair if and only if $2A \subseteq A$ (i.e., if $k \in A$ then $2k \in A$) and $B = A - 2A$.

(c) [1.5] Note that according to (b), if $A = \{1, 2, 4, 8, \ldots, 2^m, \ldots\}$ and $B = \{1\}$, then $(A, B)$ is an Euler pair. What familiar fact is this equivalent to?
(d) [3.5] Let \( c(n) \) denote the number of ways to write \( n \) as a sum (without regard to order) of positive integers, such that any two of the summands differ by at least two. Let \( d(n) \) denote the number of ways to write \( n \) as a sum (without regard to order) of positive integers of the form \( 5k − 1 \) and \( 5k + 1 \), where \( k \) is an integer. For instance, \( c(10) = 5 \), since \( 10 = 8 + 2 = 7 + 3 = 6 + 4 = 6 + 3 + 1 \); while \( d(10) = 5 \), since 
\[
1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 6 + 1 + 1 + 1 + 1 = 4 + 4 + 1 + 1 = 6 + 4.
\]
Show that \( c(n) = d(n) \) for all \( n \).

136. [2.5] Let
\[
f(n) = \sum a_1 a_2 \cdots a_k,
\]
where the sum is over all \( 2^{n-1} \) ways of writing \( n \) as an ordered sum \( a_1 + \cdots + a_k \) of positive integers \( a_i \). For instance,
\[
f(4) = 4 + 3 \cdot 1 + 1 \cdot 3 + 2 \cdot 2 + 2 \cdot 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 \cdot 1 = 21.
\]
Find a simple expression for \( f(n) \) in terms of Fibonacci numbers.

137. [2] Let \( 0^\circ \leq \theta \leq 180^\circ \) and \( 0 < t < 1 \). A person stands at the origin in the \((x, y)\)-plane and steps a distance of 1 in the positive \( x \)-direction. He then turns an angle \( \theta \) counterclockwise and steps a distance \( t \). He again turns \( \theta \) counterclockwise and steps \( t^2 \). Continuing in this way, at the \( n \)th step he turns \( \theta \) and steps a distance of \( t^{n-1} \). As \( n \) increases, he will approach a limiting point \( f(\theta, t) \) in the \((x, y)\)-plane. For instance,
\[
f(0^\circ, t) = (1 + t + t^2 + \cdots, 0) = (1/(1-t), 0)
\]
\[
f(180^\circ) = (1 - t + t^2 - t^3 + \cdots, 0) = (1/(1+t), 0).
\]
Find a simple formula for \( f(\theta, t) \).
138. [2.5] Let \( x \) be a positive real number. Find the maximum value of the product \( \prod_{i \in S} i \), where \( S \) is any subset of the positive real numbers whose sum is \( x \). (HINT: First show that if the number \( k \) of elements of \( S \) is fixed, then maximum is achieved by taking all the elements of \( S \) to be equal to \( x/k \). Then find the best value of \( k \). For most numbers, \( k \) will be unique. But for each \( k \geq 1 \), there is an exceptional number \( x_k \) such that there are two sets \( S \) and \( S' \) which achieve the maximum, one with \( k \) elements and one with \( k + 1 \) elements.)

139. [4] A polynomial \( f(x) \in \mathbb{C}[x] \) is indecomposable if whenever \( f(x) = r(s(x)) \) for polynomials \( r(x), s(x) \), then either \( \deg r(x) = 1 \) or \( \deg s(x) = 1 \). Suppose that \( f(x) \) and \( g(x) \) are nonconstant indecomposable polynomials in \( \mathbb{C}[x] \) such that \( f(x) - g(y) \) factors in \( \mathbb{C}[x, y] \). (A trivial example is \( x^2 - y^2 = (x - y)(x + y) \).) Show that either \( g(x) = f(ax + b) \) for some \( a, b \in \mathbb{C} \), or else

\[
\deg f(x) = \deg g(x) = 7, 11, 13, 15, 21, \text{ or } 31.
\]

Moreover, this result is best possible in the sense that for \( n = 7, 11, 13, 15, 21, \text{ or } 31 \), there exist indecomposable polynomials \( f(x), g(x) \) of degree \( n \) such that \( f(x) \neq g(ax + b) \) and \( f(x) - g(y) \) factors.