18.S34 (FALL 2006)
PROBLEMS ON INEQUALITIES

1. Let $a$ be a real number and $n$ a positive integer, with $a > 1$. Show that
   \[ a^n - 1 \geq n \left( a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}} \right). \]

2. Let $x_i > 0$ for $i = 1, 2, \ldots, n$. Show that
   \[ (x_1 + x_2 + \cdots + x_n) \left( \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right) \geq n^2. \]

3. If $x_i > 0$, $q_i > 0$ for $i = 1, 2, \ldots, n$, and $q_1 + \cdots + q_n = 1$, show that
   \[ x_1^{q_1} \cdots x_n^{q_n} \leq q_1 x_1 + \cdots + q_n x_n. \]

4. For $p > 1$ and $a_1, a_2, \ldots, a_n$ positive, show that
   \[ \sum_{k=1}^{n} \left( \frac{a_1 + a_2 + \cdots + a_k}{k} \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{k=1}^{n} a_k^p. \]

5. If $a_n > 0$ for $n = 1, 2, \ldots$, show that
   \[ \sum_{n=1}^{\infty} \sqrt[k]{a_1 a_2 \cdots a_n} \leq e \sum_{n=1}^{\infty} a_n, \]
   provided that $\sum_{n=1}^{\infty} a_n$ converges.

6. Let $0 < x < \pi/2$. Show that
   \[ x - \sin x \leq \frac{1}{6} x^3. \]

7. Show that
   \[ 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\sqrt{n} + 1 - 2. \]
8. Let
\[
\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}
\]
be \(n\) fractions with \(b_i > 0\) for \(i = 1, 2, \ldots, n\). Show that the fraction
\[
\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}
\]
is contained between the largest and smallest of these \(n\) fractions.

9. For \(n = 1, 2, 3, \ldots\) let
\[
x_n = \frac{1000^n}{n!}.
\]
Find the largest term of the sequence.

10. Suppose that \(a_1, a_2, \ldots, a_n\) with \(n \geq 2\) are real numbers larger than \(-1\), and moreover all \(a_j\)'s have the same sign. Show that
\[
(1 + a_1)(1 + a_2) \cdots (1 + a_n) > 1 + a_1 + a_2 + \cdots + a_n.
\]

11. Show that
\[
\frac{1 \cdot 3 \cdot 5 \cdots 2n - 1}{2 \cdot 4 \cdot 6 \cdots 2n} < \frac{1}{\sqrt{2n + 1}}.
\]

12. Prove Chebyshev's inequality: If \(a_1 \leq a_2 \leq \cdots \leq a_n\) and \(b_1 \leq b_2 \leq \cdots \leq b_n\), then
\[
\left( \frac{1}{n} \sum_{k=1}^{n} a_k \right) \left( \frac{1}{n} \sum_{k=1}^{n} b_k \right) \leq \frac{1}{n} \sum_{k=1}^{n} a_k b_k.
\]
Generalize to more than two sets of increasing sequences.

13. Let \(n\) be a positive integer larger than 1, and let \(a > 0\). Show that
\[
\frac{1 + a + a^2 + \cdots + a^n}{a + a^2 + a^3 + \cdots + a^{n-1}} \geq \frac{n + 1}{n - 1}.
\]

14. Show that if \(a > b > 0\), then \(A < B\), where
\[
A = \frac{1 + a + \cdots + a^{n-1}}{1 + a + \cdots + a^n}, \quad B = \frac{1 + b + \cdots + b^{n-1}}{1 + b + \cdots + b^n}.
\]
15. Let \( x > 0 \), and let \( n \) be a positive integer. Show that
\[
\frac{x^n}{1 + x + x^2 + \ldots + x^{2n}} \leq \frac{1}{2n + 1}.
\]

16. Let \( a, b > 0 \), \( a + b = 1 \), and \( q > 0 \). Show that
\[
(a + \frac{1}{a})^q + (b + \frac{1}{b})^q \geq \frac{5q}{2q - 1}.
\]

17. Let \( x, y > 0 \) with \( x \neq y \), and let \( m \) and \( n \) be positive integers. Show that
\[
x^m y^n + x^n y^m < x^{m+n} + y^{m+n}.
\]

18. Let \( x > 0 \) but \( x \neq 1 \), and let \( n \) be a positive integer. Show that
\[
x^{2n-1} + x < x^{2n} + 1.
\]

19. Let \( a > b > 0 \), and let \( n \) be a positive integer greater than 1. Show that
\[
\sqrt[n]{a} - \sqrt[n]{b} < \sqrt[n]{a - b}.
\]

20. Let \( a, b, x > 0 \) and \( a \neq b \). Show that
\[
\left(\frac{a + x}{b + x}\right)^{b + x} > \left(\frac{a}{b}\right)^x.
\]

21. Let \( a > b > 0 \), and let \( n \) be a positive integer greater than 1. Show that for \( k \geq 0 \),
\[
\sqrt[n]{a^n + k^n} - \sqrt[n]{b^n + k^n} \leq a - b.
\]

22. Let \( x \geq 0 \), and let \( m \) and \( n \) be real numbers such that \( m \geq n > 0 \). Show that
\[
(m + n)(1 + x^n) \geq 2n \frac{1 - x^{m+n}}{1 - x^n}.
\]

23. Let \( a_i \geq 0 \) for \( 1 \leq i \leq n \), and let \( \sum_{i=1}^{n} a_i = 1 \). Let \( 0 \leq x_i \leq 1 \) for \( 1 \leq i \leq n \). Show that
\[
\frac{a_1}{1 + x_1} + \frac{a_2}{1 + x_2} + \ldots + \frac{a_n}{1 + x_n} \leq \frac{1}{1 + x_1 a_2 x_2 \ldots x_n a_n}.
\]
24. If $a_1, \ldots, a_{n+1}$ are positive real numbers with $a_1 = a_{n+1}$, show that
\[
\sum_{i=1}^{n} \left( \frac{a_i}{a_{i+1}} \right)^n \geq \sum_{i=1}^{n} \frac{a_{i+1}}{a_i}.
\]

25. Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be two sets of real numbers with $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$. Put $s_k = a_1 + a_2 + \cdots + a_k$ for $k = 1, 2, \ldots, n$; and let $M$ and $m$ denote respectively the largest and smallest of the numbers $s_1, s_2, \ldots, s_n$. Show that
\[
mb_1 \leq \sum_{i=1}^{n} a_i b_i \leq Mb_1.
\]

26. Show that for any real numbers $a_1, a_2, \ldots, a_n$,
\[
\left( \sum_{i=1}^{n} \frac{a_i}{i} \right)^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j}{i + j - 1}.
\]

27. Let $f$ and $g$ be real-valued functions defined on the set of real numbers. Show that there are numbers $x$ and $y$ such that $0 \leq x \leq 1$, $0 \leq y \leq 1$, and
\[
|xy - f(x) - g(x)| \geq 1/4.
\]

28. Let $t > 0$. Show that
\[
t^\alpha - \alpha t \leq 1 - \alpha, \quad \text{if } 0 < \alpha < 1
\]

and
\[
t^\alpha - \alpha t \geq 1 - \alpha, \quad \text{if } \alpha > 1.
\]

29. Show that for any real number $x$ and any positive integer $n$ we have
\[
\left| \sum_{k=1}^{n} \frac{\sin kx}{k} \right| \leq 2\sqrt{n}.
\]

30. Show that if $x$ is larger than any of the numbers $a_1, a_2, \ldots, a_n$, then
\[
\frac{1}{x - a_1} + \frac{1}{x - a_2} + \cdots + \frac{1}{x - a_n} \geq \frac{n}{x - \frac{1}{n}(a_1 + a_2 + \cdots + a_n)}.
\]
31. Show that
\[ \sqrt{\binom{n}{1}} + \sqrt{\binom{n}{2}} + \cdots + \sqrt{\binom{n}{n}} \leq \sqrt{n(2^n - 1)}. \]

32. Let \( y = f(x) \) be a continuous, strictly increasing function of \( x \) for \( x \geq 0 \), with \( f(0) = 0 \), and let \( f^{-1} \) denote the inverse function to \( f \). If \( a \) and \( b \) are nonnegative constants, then show that
\[ ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy. \]

33. Show that for \( t \geq 1 \) and \( s \geq 0 \),
\[ ts \leq t \log t - t + e^s. \]

34. Let \( a_1/b_1, a_2/b_2, \ldots, \) with each \( b_i > 0 \), be a strictly increasing sequence. Let
\[ A_j = a_1 + a_2 + \cdots + a_j, \quad \text{and} \quad B_j = b_1 + b_2 + \cdots + b_j. \]
Show that the sequence \( A_1/B_1, A_2/B_2, \ldots \) is also strictly increasing.

35. Let \( m, n \) be positive integers, and let \( a_1, a_2, \ldots, a_n \) be positive real numbers. For \( i = 1, 2, 3 \ldots \) put \( a_{n+i} = a_i \) and
\[ b_i = a_{i+1} + a_{i+2} + \cdots + a_{i+m}. \]
Show that
\[ m^n a_1 a_2 \cdots a_n < b_1 b_2 \cdots b_n, \]
except if all the \( a_i \) are equal.

36. Let \( a_1, a_2, \ldots, a_n \) be real numbers. Show that
\[ \min_{a_i \neq a_j} (a_i - a_j)^2 \leq M^2 \left( a_1^2 + \cdots + a_n^2 \right), \]
where
\[ M^2 = \frac{12}{n(n^2 - 1)}. \]
37. Let $x$ and $a$ be real numbers, and let $n$ be a nonnegative integer. Show that

$$|x - a|^n |x + na| \leq (x^2 + na^2)^{(n+1)/2}.$$ 

38. Given an arbitrary finite set of $n$ pairs of positive real numbers $\{(a_i, b_i) : i = 1, 2, \ldots, n\}$, show that

$$\prod_{i=1}^{n} (xa_i + (1 - x)b_i) \leq \max \left\{ \prod_{i=1}^{n} a_i, \prod_{i=1}^{n} b_i \right\},$$

for all $x \in [0, 1]$. Equality is attained only at $x = 0$ or $x = 1$, and then if and only if

$$\left( \sum_{i=1}^{n} \frac{a_i - b_i}{a_i} \right) \left( \sum_{i=1}^{n} \frac{a_i - b_i}{b_i} \right) \geq 0.$$ 

39. Show that if $m$ and $n$ are positive integers, then the smallest of the numbers $\sqrt[n]{m}$ and $\sqrt[n]{n}$ cannot exceed $\sqrt[3]{3}$.

40. Show that if $a \geq 2$ and $x > 0$, then $a^x + a^{1/x} \leq a^{x+1/x}$, with equality holding if and only if $a = 2$ and $x = 1$.

41. Show that if $x_i \geq 0$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \frac{1}{1 + x_i} \leq 1$, then $\sum_{i=1}^{n} 2^{-x_i} \leq 1$.

42. Let $0 \leq a_i < 1$ for $i = 1, 2, \ldots, n$, and put $\sum_{i=1}^{n} a_i = A$. Show that

$$\sum_{i=1}^{n} \frac{a_i}{1 - a_i} \geq \frac{nA}{n - A},$$

with equality if and only if all the $a_i$ are equal.

43. Show that for $n \geq 2$,

$$\prod_{i=0}^{n} \left( \frac{n}{i} \right) \leq \left( \frac{2n - 2}{n - 1} \right)^{n-1}.$$ 

44. Let $b_1, \ldots, b_n$ be any rearrangement of the positive numbers $a_1, \ldots, a_n$. Show that

$$\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n} \geq n.$$
45. Given that \( \sum_{i=1}^{n} b_i = b \) with each \( b_i \) a nonnegative number, show that
\[
\sum_{j=1}^{n-1} b_j b_{j+1} \leq \frac{b^2}{4}.
\]

46. Let \( n \geq 2 \) and \( 0 < x_1 < x_2 < \cdots < x_n \leq 1 \). Show that
\[
\frac{n}{\sum_{k=1}^{n} x_k} \geq \frac{1}{\sum_{k=1}^{n} \frac{1}{1 + x_k}}.
\]

47. Let \( f \) be a continuous function on the interval \([0, 1]\) such that \( 0 < m \leq f(x) \leq M \) for all \( x \) in \([0, 1]\). Show that
\[
\left( \int_{0}^{1} \frac{dx}{f(x)} \right) \left( \int_{0}^{1} f(x)dx \right) \leq \frac{(m + M)^2}{4mM}.
\]

48. Let \( x > 0 \) and \( x \neq 1 \). Show that
\[
\frac{\log x}{x - 1} \leq \frac{1}{\sqrt{x}} \quad \text{and} \quad \frac{\log x}{x - 1} \leq \frac{1 + x^{1/3}}{x + x^{1/3}}.
\]

49. Let \( 0 < y < x \). Show that
\[
\frac{x + y}{2} > \frac{x - y}{\log x - \log y}.
\]

50. Let \( x > 0 \). Show that
\[
\frac{2}{2x + 1} < \log \left( \frac{x + 1}{x} \right) < \frac{1}{\sqrt{x^2 + x}}.
\]

51. Let \( S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \). Show that
\[
n \left\{ (1 + n)^{1/n} - 1 \right\} < S_n < n \left\{ 1 - (n + 1)^{-1/n} - \frac{1}{n + 1} \right\}.
\]
52. Let \( x > 0 \) and \( y > 0 \). Show that
\[
\frac{1 - e^{-x-y}}{(x+y)(1-e^{-x})(1-e^{-y})} - \frac{1}{xy} < \frac{1}{12}.
\]

53. Let \( a, b, c, d, e, \) and \( f \) be nonegative real numbers satisfying
\[
a + b \leq e \quad \text{and} \quad c + d \leq f.
\]
Show that
\[
\sqrt{ac} + \sqrt{bd} \leq \sqrt{ef}.
\]

54. Show that for \( x > 0 \) and \( x \neq 1 \),
\[
0 \leq \frac{x \log x}{x^2 - 1} \leq \frac{1}{2}.
\]

55. Show that for \( x > 0 \),
\[
x(2 + \cos x) > 3 \sin x.
\]

56. Show that for \( 0 < x < \pi/2 \),
\[
2 \sin x + \tan x > 3x.
\]

57. Let \( x > 0, \ x \neq 1 \), and suppose that \( n \) is a positive integer. Show that
\[
x + \frac{1}{x^n} > 2n \frac{x - 1}{x^n - 1}.
\]

58. Let \( a \) be a fixed real number such that \( 0 \leq a < 1 \), and let \( k \) be a positive integer satisfying the condition \( k > (3+a)/(1-a) \). Show that
\[
\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{nk-1} > 1 + a
\]
for any positive integer \( n \).

59. Let \( a \) and \( b \) denote real numbers, and let \( r \) satisfy \( r \geq 0 \). Show that
\[
|a + b|^r \leq c_r(|a|^r + |b|^r),
\]
where \( c_r = 1 \) for \( r \leq 1 \) and \( c_r = 2^{r-1} \) for \( r > 1 \).
60. Let $0 < b \leq a$. Show that
\[
\frac{1}{8} \frac{(a-b)^2}{a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{1}{8} \frac{(a-b)^2}{b}.
\]

61. Consider any sequence $a_1, a_2, \ldots$ of real numbers. Show that
\[
\sum_{n=1}^{\infty} a_n \leq \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \left( \frac{r_n}{n} \right)^{1/2},
\]
where
\[
r_n = \sum_{k=n}^{\infty} a_k^2.
\]
(If the left-hand side of (1) is $\infty$, then so is the right-hand side.)

62. Let $a$, $b$, and $x$ be real numbers such that $0 < a < b$ and $0 < x < 1$. Show that
\[
\left( \frac{1-x^b}{1-x^{a+b}} \right)^b > \left( \frac{1-x^a}{1-x^{a+b}} \right)^a.
\]

63. Let $0 < a < 1$. Show that
\[
\frac{2}{e} < a^{\frac{a}{a-1}} + a^{\frac{1}{a-1}} < 1.
\]

64. Let $0 < x < 2\pi$. Show that
\[
-\frac{1}{2} \tan \frac{x}{4} \leq \sum_{k=1}^{n} \sin kx \leq \frac{1}{2} \cot \frac{x}{4}.
\]

65. Let $0 < a_k < 1$ for $k = 1, 2, \ldots, n$, with $a_1 + a_2 + \cdots + a_n < 1$. Show that
\[
\frac{1}{1 - \sum_{k=1}^{n} a_k} > \prod_{k=1}^{n} (1 + a_k) > 1 + \sum_{k=1}^{n} a_k
\]
and
\[
\frac{1}{1 + \sum_{k=1}^{n} a_k} > \prod_{k=1}^{n} (1 - a_k) > 1 - \sum_{k=1}^{n} a_k
\]
66. Show that
\[
\frac{1}{(n-1)!} \int_{n}^{\infty} w(t)e^{-t}dt < \frac{1}{(e-1)^n},
\]
where \( t \) is real, \( n \) is a positive integer, and

\[ w(t) = (t-1)(t-2) \cdots (t-n+1). \]