MY PUTNAM PROBLEMS

These are the problems I proposed when I was on the Putnam Problem Committee for the 1983-86 Putnam Exams. Problems intended to be A1 or B1 (and therefore relatively easy) are marked accordingly. The problems marked with asterisks actually appeared on the Putnam Exam (possibly reworded). — R. Stanley

1. (A1 or B1 problem) Given that

\[ \int_0^1 \frac{\log(1 + x)}{x} \, dx = \frac{\pi^2}{12}, \]

evaluate

\[ \int_0^1 \int_0^y \frac{\log(1 + x)}{x} \, dx \, dy. \]

2* (A1 or B1 problem) Let \( B \) be an \( a \times b \times c \) brick. Let \( C_1 \) be the set of all points \( p \) in \( \mathbb{R}^3 \) such that the distance from \( p \) to \( C \) (i.e., the minimum distance between \( p \) and a point of \( C \)) is at most one. Find the volume of \( C_1 \).

3* (A1 or B1 problem) If \( n \) is a positive integer, then define

\[ f(n) = 1! + 2! + \cdots + n!. \]

Find polynomials \( P(n) \) and \( Q(n) \) such that

\[ f(n + 2) = P(n)f(n + 1) + Q(n)f(n), \]

for all \( n \geq 1 \).

4. (A1 or B1 problem) Let \( C \) be a circle of radius 1, and let \( D \) be a diameter of \( C \). Let \( P \) be the set of all points inside or on \( C \) such that \( p \) is closer to \( D \) than it is to the circumference of \( C \). Find a rational number \( r \) such that the area of \( P \) is \( r \).

5. Let \( n \) be a positive integer, let \( 0 \leq j < n \), and let \( f_n(j) \) be the number of subsets \( S \) of the set \( \{0, 1, \ldots, n-1\} \) such that the sum of the elements
of $S$ gives a remainder of $j$ upon division by $n$. (By convention, the sum of the elements of the empty set is 0.) Prove or disprove:

$$f_n(j) \leq f_n(0),$$

for all $n \geq 1$ and all $0 \leq j < n$.

6. Let $P$ be the set of all real polynomials all of whose coefficients are either 0 or 1. Find

$$\inf\{\alpha \in \mathbb{R} : \exists f \in P \text{ such that } f(\alpha) = 0\}$$

and

$$\sup\{\alpha \in \mathbb{R} : \exists f \in P \text{ such that } f(0) = 1 \text{ and } f(\alpha) = 0\}.$$

Here $\inf$ denotes infimum (greatest lower bound) and $\sup$ denotes supremum (least upper bound).

Somewhat more difficult:

$$\sup\{\alpha \in \mathbb{R} : \exists f \in P \text{ such that } f(i\alpha) = 0\},$$

where $i^2 = -1$.

7. Let $n$ be a positive integer, and let $X_n$ be the set of all $n \times n$ matrices whose entries are +1 or −1. Call a nonempty subset $S$ of $X_n$ full if whenever $A \in S$, then any matrix obtained from $A$ by multiplying a row or column by −1 also belongs to $S$. Let $w(A)$ denote the number of entries of $A$ equal to 1. Find, as a function of $n$,

$$\max \frac{1}{|S|} \sum_{A \in S} w(A)^3,$$

where $S$ ranges over all full subsets of $X_n$. (Express your answer as a polynomial in $n$.)

8* Let $R$ be the region consisting of all triples $(x, y, z)$ of nonnegative real numbers satisfying $x + y + z \leq 1$. Let $w = 1 - x - y - z$. Express the value of the triple integral

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^1 y^9 z^8 w^4 \, dx \, dy \, dz$$

in the form $a!b!c!d!/n!$, where $a, b, c, d,$ and $n$ are positive integers.
9. Let \( n \) be a positive integer, and let \( f(n) \) denote the last nonzero digit in the decimal expansion of \( n! \). For instance, \( f(5) = 2 \).

(a) Show that if \( a_1, a_2, \ldots, a_k \) are distinct positive integers, then \( f(5^{a_1} + 5^{a_2} + \cdots + 5^{a_k}) \) depends only on the sum \( a_1 + a_2 + \cdots + a_k \).

(b) Assuming (a), we can define \( g(s) = f(5^{a_1} + 5^{a_2} + \cdots + 5^{a_k}) \), where \( s = a_1 + a_2 + \cdots + a_k \). Find the least positive integer \( p \) for which

\[
g(s) = g(s + p), \quad \text{for all } s \geq 1,
\]

or else show that no such \( p \) exists.

10. A transversal of an \( n \times n \) matrix is a set of \( n \) entries of \( A \), no two in the same row or column. Let \( f(n) \) be the number of \( n \times n \) matrices \( A \) satisfying the following two conditions:

(a) Each entry of \( A \) is either \(-1, 1, \) or \(0\).

(b) All transversals of \( A \) have the same sum of their elements.

Find a formula for \( f(n) \) of the form

\[
a_1 \cdot b_1^n + a_2 \cdot b_2^n + a_3 \cdot b_3^n + a_4,
\]

where \( a_i, b_i \) are rational numbers.

Easier version (not on Putnam Exam):

(a) Each entry of \( A \) is either \(0 \) or \(1\).

(b) All transversals of \( A \) have the same number of \(1\)'s.

11. Let \( T \) be a triangle and \( R, S \) rectangles inscribed in \( T \) as shown: 
Find the maximum value, or show that no maximum exists, of

\[ \frac{A(R) + A(S)}{A(T)}, \]

where \( T \) ranges over all triangles and \( R, S \) over all rectangles as above, and where \( A \) denotes area.

12* (A1 or B1 problem) Inscribe a rectangle of base \( b \) and height \( h \) and an isosceles triangle of base \( b \) in a circle of radius one as shown.

For what value of \( h \) do the rectangle and triangle have the same area?

13* If \( p(x) = \sum_{i=0}^{m} a_i x^i \) is a polynomial with real coefficients \( a_i \), then set

\[ (p(x)) = \sum_{i=0}^{m} a_i^2. \]

Let \( f(x) = 3x^2 + 7x + 2 \). Find (with proof) a polynomial \( g(x) \) satisfying

\[ g(0) = 1, \text{ and } \]

\[ (f(x)^n) = (g(x)^n) \text{ for every integer } n \geq 1. \]

14* Define polynomials \( f_n(x) \) for \( n \geq 0 \) by

\[
\begin{align*}
    f_0(x) &= 1 \\
    f_{n+1}'(x) &= (n + 1)f_n(x + 1), \quad n \geq 0 \\
    f_n(0) &= 0, \quad n \geq 1.
\end{align*}
\]
Find (with proof) the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

Variation (not on Putnam Exam): $f_0(x) = 1$, $f_{n+1}(x) = xf_n(x) + f'_n(x)$. Find $f_{2n}(0)$.

15. Define

$$c(k, n) = \cos \frac{\pi k}{n} + \sqrt{1 + \cos^2 \frac{\pi k}{n}}.$$

Find (with proof) all positive integers $n$ satisfying

$$c(1, n) = c(2, n)c(3, n).$$

16. Let $R$ be a ring (not necessarily with identity). Suppose that there exists a nonzero element $x$ of $R$ satisfying

$$x^4 + x = 2x^3.$$

Prove or disprove: There exists a nonzero element $y$ of $R$ satisfying $y^2 = y$.

17. Find the largest real number $\lambda$ for which there exists a $10 \times 10$ matrix $A = (a_{ij})$, with each entry $a_{ij}$ equal to 0 or 1, and with exactly 84 0’s, and there exists a nonzero column vector $x$ of length 10 with real entries, such that $Ax = \lambda x$.

18. Choose two points $p$ and $q$ independently and uniformly from the square $0 \leq x \leq 1, 0 \leq y \leq 1$ in the $(x, y)$-plane. What is the probability that there exists a circle $C$ contained entirely within the first quadrant $x \geq 0, y \geq 0$ such that $C$ contains $x$ and $y$ in its interior? Express your answer in the form $1 - (a + b\pi)(c + d\sqrt{e})$ for rational numbers $a, b, c, d, e$.

19.* (A1 or B1 problem) Let $k$ be the smallest positive integer with the following property:

There are distinct integers $m_1, m_2, m_3, m_4, m_5$ such that the polynomial $p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$ has exactly $k$ nonzero coefficients.
Find, with proof, a set of integers \( m_1, m_2, m_3, m_4, m_5 \) for which this minimum \( k \) is achieved.

NOTE. The original version of this problem was considerably more difficult (and was not intended for A1 or B1). It was as follows:

Let \( P(x) = x^{11} + a_{10}x^{10} + \cdots + a_0 \) be a monic polynomial of degree eleven with real coefficients \( a_i \), with \( a_0 \neq 0 \). Suppose that all the zeros of \( P(x) \) are real, i.e., if \( \alpha \) is a complex number such that \( P(\alpha) = 0 \), then \( \alpha \) is real. Find (with proof) the least possible number of nonzero coefficients of \( P(x) \) (including the coefficient 1 of \( x^{11} \)).

20. Find (with proof) the largest integer \( k \) for which there exist three 9-element subsets \( X_1, X_2, X_3 \) of real numbers and \( k \) triples \( (a_1, a_2, a_3) \) satisfying \( a_i \in X_i \) and \( a_1 + a_2 + a_3 = 0 \).

21. Let

\[
S = \sum \frac{1}{m^2n^2},
\]

where the sum ranges over all pairs \( (m, n) \) of positive integers such that the largest power of 2 dividing \( m \) is different from the largest power of 2 dividing \( n \). Express \( S \) in the form \( \alpha \pi^k \), where \( k \) is an integer and \( \alpha \) is rational. You may assume the formula

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]

22. Let \( a \) and \( b \) be nonnegative integers with binary expansions \( a = a_0 + 2a_1 + \cdots \) and \( b = b_0 + 2b_1 + \cdots \) (so \( a_i, b_i = 0 \) or 1), and define

\[
a \land b = a_0b_0 + 2a_1b_1 + 4a_2b_2 + \cdots = \sum 2^i a_i b_i.
\]

Given an integer \( n \geq 0 \), define \( f(n) \) to be the number of pairs \( (a, b) \) of nonnegative integers satisfying \( n = a + b + (a \land b) \). Find a polynomial \( P(x) \) for which

\[
\sum_{n=0}^{\infty} f(n)x^n = \prod_{k=0}^{\infty} P\left(x^{2^k}\right), \quad |x| < 1,
\]

or show that no such \( P(x) \) exists.
23. Given \( v = (v_1, \ldots, v_n) \) where each \( v_i = 0 \) or 1, let \( f(v) \) be the number of even numbers among the \( n \) numbers

\[
v_1 + v_2 + v_3 + v_2 + v_3 + v_4 + \ldots, v_{n-2} + v_{n-1} + v_n, v_{n-1} + v_n + v_1, v_n + v_1 + v_2.
\]

For which positive integers \( n \) is the following true: for all \( 0 \leq k \leq n \), exactly \( \binom{n}{k} \) vectors of the \( 2^n \) vectors \( v \in \{0, 1\}^n \) satisfy \( f(v) = k \)?

24. Let \( p \) be a prime number. Let \( c_k \) denote the coefficient of \( x^{2k} \) in the polynomial \((1 + x + x^3 + x^4)^k\). Find the remainder when the number \( \sum_{k=0}^{n-1}(-1)^k c_k \) is divided by \( p \). Your answer should depend only on the remainder obtained when \( p \) is divided by some fixed number \( n \) (independent of \( p \)).

25. Let \( x(t) \) and \( y(t) \) be real-valued functions of the real variable \( t \) satisfying the differential equations

\[
\frac{dx}{dt} = -xt + 3xy - 2t^2 + 1
\]
\[
\frac{dy}{dt} = xt + yt + 2t^2 - 1,
\]

with the initial conditions \( x(0) = y(0) = 1 \). Find \( x(1) + 3y(1) \). (This problem was later withdrawn for having an easier than intended solution.)

26. Let \( a_1, \ldots, a_n, b_1, \ldots, b_n \) be real numbers with \( 1 \leq b_1 < b_2 < \cdots < b_n \). Suppose that there is a polynomial \( f(x) \) satisfying

\[
(1 - x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.
\]

Express \( f(1) \) in terms of \( b_1, \ldots, b_n \) and \( n \) (but independent from \( a_1, \ldots, a_n \)).

27. Given positive integers \( n \) and \( i \), let \( x \) be the unique real number \( \geq i \) satisfying \( x^{i-1} = n \). Define \( f(n, i) = (x + 1)^{x-i} \), and set \( f(0, i) = 0 \) for all \( i \). Suppose that \( a_1, a_2, \ldots \) is a nonnegative integer sequence satisfying \( a_{i+1} \leq f(a_i, i) \) for all \( i \geq 1 \). Prove or disprove: \( a_i \) is a polynomial function of \( i \) for \( i \) sufficiently large.
28. Let $0 \leq x \leq 1$. Let the binary expansion of $x$ be
\[ x = a_1 2^{-1} + a_2 2^{-2} + \cdots \]
(where, say, we never choose the expansion ending in infinitely many 1’s). Define
\[ f(x) = a_1 3^{-1} + a_2 3^{-2} + \cdots . \]
In other words, write $x$ in binary and read $x$ in ternary. Evaluate $\int_0^1 f(x)dx$.

29* Let $f(x, y, z) = x^2 + y^2 + z^2 + xyz$. Let $p(x, y, z)$, $q(x, y, z)$, and $r(x, y, z)$ be polynomials satisfying
\[ f(p(x, y, z), q(x, y, z), r(x, y, z)) = f(x, y, z). \]
Prove or disprove: $(p, q, r)$ consists of some permutation of $(\pm x, \pm y, \pm z)$, where the number of minus signs is even.

30. Let
\[ \frac{1}{1 - x - y - z - 6(xy + xz + yz)} = \sum_{r,s,t=0}^{\infty} f(r, s, t)x^ry^sz^t \]
(convergent for $|x|, |y|, |z|$ sufficiently small). Find the largest real number $R$ for which the power series
\[ F(u) = \sum_{n=0}^{\infty} f(n, n, n)u^n \]
converges for all $|u| < R$. 
