18.S66 PROBLEMS #2

Spring 2003

35. ($*$) Let $f(n)$ denote the number of subsets of $\mathbb{Z}/n\mathbb{Z}$ (the integers modulo n) whose elements sum to 0 (mod n) (including the empty set $\emptyset$). For instance, $f(5) = 8$, corresponding to $\emptyset$, $\{0\}$, $\{1,4\}$, $\{0,1,4\}$, $\{2,3\}$, $\{0,2,3\}$, $\{1,2,3,4\}$, $\{0,1,2,3,4\}$. When n is odd, $f(n)$ is equal to the number of "necklaces" (up to cyclic rotation) with n beads, each bead colored white or black. For instance, when $n = 5$ the necklaces are (writing 0 for white and 1 for black) 00000, 00001, 00011, 00101, 00111, 01011, 01111, 11111. (This is easy if n is prime.)

36. In how many ways can n square envelopes of different sizes by arranged by inclusion? For instance, with six envelopes A, B, C, D, E, F (listed in decreasing order of size), one way of arranging them would be $F \in C \in B, E \in B, D \in A$, where $I \in J$ means that envelope I is contained in envelope J.

37. Let $w = a_1 a_2 \cdots a_n$ be a permutation of $1, 2, \ldots, n$, denoted $w \in \mathfrak{S}_n$. We can also regard w as the bijection $w : [n] \to [n]$ defined by $w(i) = a_i$. We say that i is a *fixed point* of w if $w(i) = i$ (or $a_i = i$). The total number of fixed points of all $w \in \mathfrak{S}_n$ is $n!$.

38. An *inversion* of w is a pair (i,j) for which $i < j$ and $a_i > a_j$. Let $\mathrm{inv}(w)$ denote the number of inversions of w. Then

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

39. For any $w \in \mathfrak{S}_n$, $\mathrm{inv}(w) = \mathrm{inv}(w^{-1})$.

40. How many permutations $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ have the property that for all $1 \le i < n$, the numbers appearing in w between i and $i+1$ (whether i is to the left or right of $i+1$) are all less than i? An example of such a permutation is 976412358.

41. How many permutations $a_1a_2\cdots a_n \in \mathfrak{S}_n$ satisfy the following property: if $2 \leq j \leq n$, then $|a_i - a_j| = 1$ for some $1 \leq i < j$? E.g., for $n = 3$ there are the four permutations 123, 213, 231, 321.

42. A *derangement* is a permutation with no fixed points. Let $D(n)$ denote the number of derangments of $[n]$ (i.e., the number of $w \in \mathfrak{S}_n$ with no fixed points). (Set $D(0) = 1$.) Show that

$$D(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}\right). \tag{2}$$

NOTE. A rather complicated recursive bijection follows from a general technique for converting Inclusion-Exclusion arguments to bijective proofs. It would be nice, however, to have a "direct" proof of the identity

$$D(n) + \frac{n!}{1!} + \frac{n!}{3!} + \cdots = n! + \frac{n!}{2!} + \frac{n!}{4!} + \cdots.$$

In other words, the number of ways to choose a permutation $w \in \mathfrak{S}_n$ and then choose an odd number of fixed points of w, or instead to choose a derangement in $\mathfrak{S}_n$, is equal to the number of ways to choose $w \in \mathfrak{S}_n$ and then choose an even number of fixed points of w.

43. Show that

$$D(n) = (n-1)(D(n-1) + D(n-2)), \quad n \geq 1.$$

44. Show that

$$D(n) = nD(n-1) + (-1)^n.$$

(Trivial from (2), but surprisingly tricky to do bijectively.)

45. Let $m_1, \ldots, m_n \in \mathbb{N}$ and $\sum im_i = n$. Show that the number of $w \in \mathfrak{S}_n$ whose disjoint cycle decomposition contains exactly m_i cycles of length i is equal to

$$\frac{n!}{1^{m_1}\, m_1!\, 2^{m_2}\, m_2! \cdots n^{m_n}\, m_n!}.$$

Note that, contrary to certain authors, we are including cycles of length one (fixed points).

46. A *fixed point free involution* in $\mathfrak{S}_{2n}$ is a permutation $w \in \mathfrak{S}_{2n}$ satisfying $w^2 = 1$ and $w(i) \neq i$ for all $i \in [2n]$. The number of fixed point free involutions in $\mathfrak{S}_{2n}$ is $(2n-1)!! := 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

    NOTE. This problem is a special case of Problem 45. For the present problem, however, give a factor-by-factor explanation of the product $1 \cdot 3 \cdot 5 \cdots (2n-1)$.

47. If $X \subseteq \mathbb{P}$, then write $-X = \{-n : n \in X\}$. Let $g(n)$ be the number of ways to choose a subset X of $[n]$, and then choose fixed point free involutions π on $X \cup (-X)$ and $\bar{\pi}$ on $\bar{X} \cup (-\bar{X})$, where $\bar{X} = \{i \in [n] : i \notin X\}$. Then $g(n) = 2^n\, n!$.

48. Let $n \geq 2$. The number of permutations $w \in \mathfrak{S}_n$ with an even number of even cycles (in the disjoint cycle decomposition of w) is $n!/2$.

49. Let $c(n,k)$ denote the number of $w \in \mathfrak{S}_n$ with k cycles (in the disjoint cycle decomposition of w). Show that

$$\sum_{k=1}^{n} c(n,k)x^k = x(x+1)(x+2)\cdots(x+n-1).$$

    Try to give *two* bijective proofs, viz., first letting $x \in \mathbb{P}$ and showing that both sides are equal as integers, and second by showing that the coefficients of x^k on both sides are equal.

50. Let w be a random permutation of $1, 2, \ldots, n$ (chosen from the uniform distribution). Fix a positive integer $1 \leq k \leq n$. What is the probability that in the disjoint cycle decomposition of w, the length of the cycle containing 1 is k? In other words, what is the probability that k is the least positive integer for which $w^k(1) = 1$?

    NOTE. Let p_{nk} be the desired probability. Then $p_{nk} = f_{nk}/n!$, where f_{nk} is the number of $w \in \mathfrak{S}_n$ for which the length of the cycle containing 1 is k. Hence one needs to determine the number f_{nk} by a bijective argument.

51. A *record* (or *left-to-right maximum*) of a permutation $a_1 a_2 \cdots a_n$ is a term a_j such that $a_j > a_i$ for all $i < j$. The number of $w \in \mathfrak{S}_n$ with k records equals the number of $w \in \mathfrak{S}_n$ with k cycles.

52. (?) Let $a(n)$ be the number of permutations $w \in \mathfrak{S}_n$ that have a square root, i.e., there exists $u \in \mathfrak{S}_n$ satisfying $u^2 = w$. Then $a(2n+1) = (2n+1)a(2n)$. (This might be easy.)

53. Let $w = a_1 \cdots a_n \in \mathfrak{S}_n$. An *excedance* of w is a number i for which $a_i > i$. A *descent* of w is a number i for which $a_i > a_{i+1}$. Show that the number of $w \in \mathfrak{S}_n$ with k excedances is equal to the number of $w \in \mathfrak{S}_n$ with k descents. (This number is denoted $A(n, k+1)$ and is called an *Eulerian number*.)

54. Continuing the previous problem, a *weak excedance* of w is a number i for which $a_i \geq i$. Show that the number of $w \in \mathfrak{S}_n$ with k weak excedances is equal to $A(n, k)$ (the number of $w \in \mathfrak{S}_n$ with $k-1$ excedances).

55. Let $i_1, \ldots, i_k \in \mathbb{N}$, $\sum i_j = n$. The *multinomial coefficient* $\binom{n}{i_1,\ldots,i_k}$ is defined combinatorially to be the number of permutations of the multiset $\{1^{i_1}, \ldots, k^{i_k}\}$. For instance, $\binom{4}{1,2,1} = 12$, corresponding to the twelve permutations 1223, 1232, 1322, 2123, 2132, 2213, 2231, 2312, 2321, 3122, 3212, 3211. Then

$$\binom{n}{i_1, \ldots, i_k} = \frac{n!}{i_1! \cdots i_k!}.$$

56. The *descent set* $D(w)$ of $w \in \mathfrak{S}_n$ is the set of descents of w. E.g., $D(47516823) = \{2, 3, 6\}$. Let $S = \{b_1, \ldots, b_{k-1}\} \subseteq [n-1]$, with $b_1 < b_2 < \cdots < b_{k-1}$. Let

$$\alpha_n(S) = \#\{w \in \mathfrak{S}_n \,:\, D(w) \subseteq S\}.$$

Then

$$\alpha_n(S) = \binom{n}{b_1, b_2 - b_1, b_3 - b_2, \ldots, b_{k-1} - b_{k-2}, n - b_{k-1}}.$$

57. The *major index* $\text{maj}(w)$ of a permutation $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ is defined by

$$\text{maj}(w) = \sum_{i \,:\, a_i > a_{i+1}} i = \sum_{i \in D(w)} i.$$

For instance, $\mathrm{maj}(47516823) = 2 + 3 + 6 = 11$. Then

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)}.$$

58. Extending the previous problem, fix j, k, n. Then

$$\begin{aligned} &\#\{w \in \mathfrak{S}_n \,:\, \mathrm{inv}(w) = j,\ \mathrm{maj}(w) = k\} \\ &\quad = \#\{w \in \mathfrak{S}_n \,:\, \mathrm{inv}(w) = k,\ \mathrm{maj}(w) = j\}. \end{aligned}$$

59. A permutation $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ is *alternating* if $D(w) = \{1, 3, 5, \ldots\} \cap [n]$. In other words,

$$a_1 > a_2 < a_3 > a_4 < a_5 > \cdots.$$

Let E_n denote the number of alternating permutations in $\mathfrak{S}_n$. Then $E_0 = E_1 = 1$ and

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}, \quad n \geq 1. \tag{3}$$

60. Show that

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x. \tag{4}$$

Note. It is not difficult to deduce this result from equation (3), but a combinatorial proof is wanted. This is quite a bit more difficult. Note that $\sec x$ is an even function of x and $\tan x$ is odd, so (4) is equivalent to

$$\begin{aligned} \sum_{n \geq 0} E_{2n} \frac{x^{2n}}{(2n)!} &= \sec x \\ \sum_{n \geq 0} E_{2n+1} \frac{x^{2n+1}}{(2n+1)!} &= \tan x. \end{aligned}$$

Note. We could actually use equation (4) to *define* $\tan x$ and $\sec x$ (and hence the other trigonometric functions in terms of these) combinatorially! The next two exercises deal with this subject of "combinatorial trigonometry."

61. Assuming (4), show that

$$1 + \tan^2 x = \sec^2 x.$$

62. Assuming (4), show that

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - (\tan x)(\tan y)}.$$

63. Let $k \geq 2$. The number of permutations $w \in \mathfrak{S}_n$ all of whose cycle lengths are divisible by k is given by

$$1^2 \cdot 2 \cdot 3 \cdots (k-1)(k+1)^2(k+2)\cdots(2k-1)(2k+1)^2(2k+2)\cdots(n-1).$$

64. Let $k \geq 2$. The number of permutations $w \in \mathfrak{S}_n$ none of whose cycle lengths is divisible by k is given by

$$1 \cdot 2 \cdots (k-1)^2(k+1)\cdots(2k-2)(2k-1)^2(2k+1)\cdots(n-1)n, \quad \text{if } k \nmid n$$

$$1 \cdot 2 \cdots (k-1)^2(k+1)\cdots(2k-2)(2k-1)^2(2k+1)\cdots(n-2)(n-1)^2, \quad \text{if } k | n.$$

65. The number of pairs $(u, v) \in \mathfrak{S}_n^2$ such that $uv = vu$ is given by $p(n)n!$, where $p(n)$ denotes the number of partitions of n.

NOTE (for those familiar with groups). This problem generalizes as follows. Let G be a finite group. The number of pairs $(u, v) \in G \times G$ such that $uv = vu$ is given by $k(G) \cdot |G|$, where $k(G)$ denotes the number of conjugacy classes of G. In this case a bijective proof is unknown (and probably impossible).

66. The number of pairs $(u, v) \in \mathfrak{S}_n^2$ such that $u^2 = v^2$ is given by $p(n)n!$ (as in the previous problem).

NOTE. Again there is a generalization to arbitrary finite groups G. Namely, the number of pairs $(u, v) \in G \times G$ such that $uv = vu$ is given by $\iota(C) \cdot |G|$, where $\iota(G)$ denotes the number of *self-inverse* conjugacy classes K of G, i.e, if $w \in K$ then $w^{-1} \in K$.

67. (*) The number of triples $(u, v, w) \in \mathfrak{S}_n^3$ such that u, v, and w are n-cycles and $uvw = 1$ is equal to 0 if n is even (this part is easy), and to $2(n-1)!^2/(n+1)$ if n is odd.

68. (*) Let n be an odd positive integer. The number of ways to write the n-cycle $(1, 2, \ldots, n) \in \mathfrak{S}_n$ in the form $uvu^{-1}v^{-1}$ $(u, v \in \mathfrak{S}_n)$ is equal to $2n \cdot n!/(n+1)$.