18.S66 PROBLEMS #4
Spring 2003

A tree $T$ on $[n]$ is a graph with vertex set $[n]$ which is connected and contains no cycles. Equivalently, as is easy to see, $T$ is connected and has $n - 1$ edges. A forest is a graph for which every connected component is a tree. A rooted tree is a tree with a distinguished vertex $u$, called the root. If there are $t(n)$ trees on $[n]$ and $r(n)$ rooted trees, then $r(n) = nt(n)$ since there are $n$ choices for the root $u$. A planted forest (sometimes called a rooted forest) is a graph for which every connected component is a rooted tree.

106. [2.5] The number of trees $t(n)$ on $[n]$ is $t(n) = n^{n-2}$. Hence the number of rooted trees is $r(n) = n^{n-1}$.

107. [1] The number of planted forests on $[n]$ is $(n+1)^{n-1}$.

108. [2] Let $S \subseteq [n]$, $\#S = k$. The number $p_S(n)$ of planted forests on $[n]$ whose root set is $S$ is given by

$$p_S(n) = kn^{n-k-1}.$$ 

109. [2] Given a planted forest $F$ on $[n]$, let $\deg(i)$ be the degree (number of children of $i$). E.g., $\deg(i) = 0$ if and only if $i$ is a leaf (endpoint) of $F$. If $F$ has $k$ components then it is easy to see that $\sum_i \deg(i) = n - k$. Given $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{N}^n$ with $\sum \delta_i = n - k$, let $N(\delta)$ denote the number of planted forests $F$ on $[n]$ (necessarily with $k$ components) such that $\deg(i) = \delta_i$ for $1 \leq i \leq n$. Then

$$N(\delta) = \binom{n-1}{k-1} \binom{n-k}{\delta_1, \ldots, \delta_n},$$

where $\binom{n-k}{\delta_1, \ldots, \delta_n}$ denotes a multinomial coefficient.

110. [2] The number of trees with $n+1$ unlabelled vertices and $n$ labelled edges is $(n+1)^{n-2}$.

111. [2.5] A $k$-edge colored tree is a tree whose edges are colored from a set of $k$ colors such that any two edges with a common vertex have
different colors. Show that the number $T_k(n)$ of $k$-edge colored trees on the vertex set $[n]$ is given by

$$T_k(n) = k(nk - n)(nk - n - 1) \cdots (nk - 2n + 3) = k(n - 2)! \binom{nk - n}{n - 2}.$$

(This problem has received little attention and may be easy.)

112. A binary tree is a rooted tree such that every vertex $v$ has exactly two subtrees $L_v, R_v$, possibly empty, and the set $\{L_v, R_v\}$ is linearly ordered, say as $(L_v, R_v)$. We call $L_v$ the left subtree of $v$ and draw it to the left of $v$. Similarly $R_v$ is called the right subtree of $v$, etc. A binary tree on the vertex set $[n]$ is increasing if each vertex is smaller that its children. An example of such a tree is given by:

![Binary Tree Diagram]

(a) [1] The number of increasing binary trees on $[n]$ is $n!$.

(b) [2] The number of increasing binary trees on $[n]$ for which exactly $k$ vertices have a left child is the Eulerian number $A(n, k + 1)$.

113. An increasing forest is a planted forest on $[n]$ such that every vertex is smaller than its children.

(a) [1] The number of increasing forests on $[n]$ is $n!$.

(b) [2] The number of increasing forests on $[n]$ with exactly $k$ components is equal to the number of permutations $w \in \mathcal{S}_n$ with $k$ cycles.

(c) [2] The number of increasing forests on $[n]$ with exactly $k$ endpoints is the Eulerian number $A(n, k)$.  

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114. [2] Show that

\[ \sum_{n \geq 0} (n+1) \frac{x^n}{n!} = \left( \sum_{n \geq 0} \frac{x^n}{n!} \right) \left( \sum_{n \geq 0} (n+1)^{-1} \frac{x^n}{n!} \right). \]

115. [2] Show that

\[ \frac{1}{1 - \sum_{n \geq 1} \frac{n^{n-1} x^n}{n!}} = \sum_{n \geq 0} \frac{n^n x^n}{n!}. \]

116. [3] Let \( \tau \) be a rooted tree with vertex set \([n]\) and root 1. An inversion of \( \tau \) is a pair \((i, j)\) such that \(1 < i < j\) and the unique path in \( \tau \) from 1 to \( i \) passes through \( j \). For instance, the tree below has the inversions \((3, 4), (2, 4), (2, 6), \) and \((5, 6)\).

![Tree](image)

Let \( \text{inv}(\tau) \) denote the number of inversions of \( \tau \). Define

\[ I_n(t) = \sum_{\tau} t^{\text{inv}(\tau)}, \]

summed over all \( n^{n-2} \) trees on \([n]\) with root 1. For instance,

\[
\begin{align*}
I_1(t) &= 1 \\
I_2(t) &= 1 \\
I_3(t) &= 2 + t \\
I_4(t) &= 6 + 6t + 3t^2 + t^3 \\
I_5(t) &= 24 + 36t + 30t^2 + 20t^3 + 10t^4 + 4t^5 + t^6 \\
I_6(t) &= 120 + 240t + 270t^2 + 240t^3 + 180t^4 + 120t^5 + 70t^6 + 35t^7 + 15t^8 + 5t^9 + t^{10}.
\end{align*}
\]
Show that
\[ t^{n-1} I_n(1 + t) = \sum_{G} t^{e(G)}, \]
summed over all connected graphs \( G \) (without loops or multiple edges) on the vertex set \([n]\), where \( e(G) \) is the number of edges of \( G \).

117. (*) An alternating tree on \([n + 1]\) is a tree with vertex set \([n + 1]\) such that every vertex is either less than all its neighbors or greater than all its neighbors. Let \( f(n) \) denote the number of alternating trees on \([n + 1]\), so \( f(1) = 1, f(2) = 2, f(3) = 7, f(4) = 36, \) etc. Then
\[ f(n) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (k + 1)^{n-1}. \]

118. [2.5] A local binary search tree is a binary tree, say with vertex set \([n]\), such that the left child of a vertex is smaller than its parent, and the right child of a vertex is larger than its parent. An example of such a tree is:

```
      5
     / \  
    4   6  
  / \  / \  
 2  7 6 8 9
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The number \( f(n) \) of alternating trees on \([n]\) is equal to the number of local binary search trees on \([n + 1]\).

119. (*) A tournament is a directed graph with no loops (edges from a vertex to itself) and with exactly one edge \( u \rightarrow v \) or \( v \rightarrow u \) between any two distinct vertices \( u, v \). Thus the number of tournaments on \([n]\) (i.e., with vertex set \([n]\)) is \( 2^{\binom{n}{2}} \). Write \( C = (c_1, c_2, \ldots, c_k) \) for the directed cycle with edges \( c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_k \rightarrow c_1 \) in a tournament on \([n]\). Let \( \text{asc}(C) \) be the number of integers \( 1 \leq i \leq k \) for which \( c_{i-1} < c_i \), and let \( \text{des}(C) \) be the number of integers \( 1 \leq i \leq k \) for
which \( c_{i-1} > c_i \), where by convention \( c_0 = c_k \). We say that the cycle \( C \) is \textit{ascending} if \( \text{asc}(C) \geq \text{des}(C) \). For example, the cycles \((a, b, c), (a, c, b, d), (a, b, d, c), \) and \((a, c, d, b)\) are ascending, where \( a < b < c < d \).

A tournament \( T \) on \([n]\) is \textit{semicyclic} if it contains no ascending cycles, i.e., if for any directed cycle \( C \) in \( T \) we have \( \text{asc}(C) < \text{des}(C) \). The number of semicyclic tournaments on \([n]\) is equal to the number of alternating trees on \([n]\). (This problem, usually stated in a different but equivalent form, has received a lot of attention. A solution would be well worth publishing.)

120. [2] An \textit{edge-labelled alternating tree} is a tree, say with \( n+1 \) vertices, whose edges are labelled \( 1, 2, \ldots, n \) such that no path contains three consecutive edges whose labels are increasing. (The vertices are not labelled.) If \( n > 1 \), then the number of such trees is \( n!/2 \).

121. [2.5] A \textit{recursively labelled tree} is a rooted tree on the vertex set \([n]\), such every subtree (i.e., every vertex and its descendants) consists of consecutive integers. An example is:

![Diagram of a recursively labelled tree]

Similarly define a \textit{recursively labelled forest}. Let \( t_n \) (respectively, \( f_n \)) denote the number of recursively labelled trees (respectively, forests) on the vertex set \([n]\). Then \( t_n \) is the number of ordered pairs of ternary trees with a total of \( n-1 \) vertices. (A \textit{ternary tree} is a rooted unlabelled tree such that every vertex has three subtrees, which may be empty, and these subtrees are linearly ordered.) Similarly \( f_n \) is the number of ternary trees with \( n \) vertices.
\textbf{NOTE.} It is known that
\[ t_n = \frac{1}{n} \binom{3n - 2}{n - 1}, \quad f_n = \frac{1}{2n + 1} \binom{3n}{n}, \]
though these formulas are not relevant to finding a bijective proof.

122. [2] A tree on a linearly ordered vertex set is called \textit{noncrossing} if \(ik\) and \(jl\) are not both edges whenever \(i < j < k < l\). The number of noncrossing trees on \([n]\) is equal to the number of ternary trees with \(n - 1\) vertices.

123. [2] A \textit{spanning tree} of a graph \(G\) is a subgraph of \(G\) which is a tree and which uses every vertex of \(G\). The number of spanning trees of \(G\) is denoted \(c(G)\) and is called the \textit{complexity} of \(G\). Thus Problem 106 is equivalent to the statement that \(c(K_n) = n^{n-2}\), where \(K_n\) is the complete graph on \(n\) vertices (one edge between every two distinct vertices). The \textit{complete bipartite graph} \(K_{mn}\) has vertex set \(A \cup B\), where \#\(A = m\) and \#\(B = n\), with an edge between every vertex of \(A\) and every vertex of \(B\) (so \(mn\) edges in all). Then \(c(K_{mn}) = m^{n-1}n^{m-1}\).

124. (*) The \textit{n-cube} \(C_n\) (as a graph) is the graph with vertex set \(\{0, 1\}^n\) (i.e., all binary \(n\)-tuples), with an edge between \(u\) and \(v\) if they differ in exactly one coordinate. Thus \(C_n\) has \(2^n\) vertices and \(n2^{n-1}\) edges. Then
\[ c(C_n) = 2^{2^{n-1} - 1} \prod_{k=1}^{n} k^\binom{n}{k}. \]

125. [2.5] A \textit{parking function} of length \(n\) is a sequence \((a_1, \ldots, a_n) \in \mathbb{P}^n\) such that its increasing rearrangement \(b_1 \leq b_2 \leq \cdots \leq b_n\) satisfies \(b_i \leq i\). The parking functions of length three are 111, 112, 121, 211, 122, 212, 221, 113, 131, 311, 123, 132, 213, 231, 312, 321. The number of parking functions of length \(n\) is \((n + 1)^{n-1}\).

126. [3] Let \(PF(n)\) denote the set of parking functions of length \(n\). Then
\[ \sum_{(a_1, \ldots, a_n) \in PF(n)} q^{a_1 + \cdots + a_n} = \sum_{\tau} q^{\binom{n+1}{2} - \text{inv}(\tau)}, \]
where \(\tau\) ranges over trees on \([n + 1]\) with root 1, and where \(\text{inv}(\tau)\) is defined in Problem 116.
127. [2.5] A valid $n$-pair consists of a permutation $w = a_1 \cdots a_n \in \mathfrak{S}_n$, together with a collection $I$ of pairs $(i, j)$ such that

- If $(i, j) \in I$ then $1 \leq i < j \leq n$.
- If $(i, j) \in I$ then $a_i < a_j$.
- If $(i, j), (i', j') \in I$ and \{\(i, i + 1, \ldots, j\) \} \subseteq \{i', i' + 1, \ldots, j'\}, then $(i, j) = (i', j')$.

For example, let $n = 3$. For each $w \in \mathfrak{S}_3$ we put after it the number of sets $I$ for which $(w, I)$ is a valid 3-pair: 123 (5), 213 (3), 132 (3), 231 (2), 312 (2), 321 (1). The number of valid $n$-pairs is $(n + 1)^{n-1}$.

128. (a) [3] Let $T$ be a tournament on $[n]$, as defined in Problem 119. The outdegree of vertex $i$, denoted $\outdeg(i)$, is the number of edges pointing out of $i$, i.e., edges of the form $i \to j$. The outdegree sequence of $T$ is defined by

$$\outdeg(T) = (\outdeg(1), \ldots, \outdeg(n)).$$

For instance, there are eight tournaments on $[3]$, but two have outdegree sequence $(1, 1, 1)$. The other six have distinct outdegree sequences, so the total number of distinct outdegree sequences of tournaments on $[3]$ is 7. The total number of distinct outdegree sequences of tournaments on $[n]$ is equal to the number of forests on $[n]$.

(b) [3] More generally, let $G$ be an (undirected) graph on $[n]$. An orientation $\mathfrak{o}$ of $G$ is an assignment of a direction $u \to v$ or $v \to u$ to each edge $uw$ of $G$. The outdegree sequence of $\mathfrak{o}$ is defined analogously to that of tournaments. The number of distinct outdegree sequences of orientations of $G$ is equal to the number of spanning forests of $G$.

129. (*) Let $G$ be a graph on $[n]$. The degree of vertex $i$, denoted $\deg(i)$, is the number of edges incident to $i$. The (ordered) degree sequence of $G$ is the sequence $(\deg(1), \ldots, \deg(n))$. The number $f(n)$ of distinct degree sequences of simple (i.e., no loops or multiple edges) graphs on $[n]$ is given by

$$f(n) = \sum_{Q} \max\{1, 2^{d(Q)-1}\},$$

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where $Q$ ranges over all graphs on $[n]$ for which every connected component is either a tree or has exactly one cycle, which is of odd length. Moreover, $d(Q)$ denotes the number of (odd) cycles in $Q$.

130. [3] The number of ways to write the cycle $(1, 2, \ldots, n) \in \mathfrak{S}_n$ as a product of $n - 1$ transpositions (the minimum possible) is $n^{n-2}$. (A \textit{transposition} is a permutation $w \in \mathfrak{S}_n$ with one cycle of length two and $n - 2$ fixed points.) For instance, the three ways to write $(1, 2, 3)$ are (multiplying right-to-left) $(1, 2)(2, 3)$, $(2, 3)(1, 3)$, and $(1, 3)(1, 2)$.

\textbf{NOTE.} It is not difficult to show bijectively that the number of ways to write \textit{some} $n$-cycle as a product of $n - 1$ transpositions is $(n - 1)! n^{n-2}$, from which the above result follows by “symmetry.” However, a direct bijection between factorizations of a \textit{fixed} $n$-cycle such as $(1, 2, \ldots, n)$ and labelled trees (say) is considerably more difficult.

131. [3.5] Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of $n$ with $\lambda_\ell > 0$, and let $w$ be a permutation of $1, 2, \ldots, n$ whose cycles have lengths $\lambda_1, \ldots, \lambda_\ell$. Let $f(\lambda)$ be the number of ways to write $w = t_1 t_2 \cdots t_k$ where the $t_i$’s are transpositions that generate all of $\mathfrak{S}_n$, and where $k$ is minimal with respect to the condition on the $t_i$’s. (It is not hard to see that $k = n + \ell - 2$.) Show that

$$f(\lambda) = (n + \ell - 2)! n^{\ell-3} \prod_{i=1}^{\ell} \frac{\lambda_i^{\lambda_i+1}}{\lambda_i!}.$$ 

\textbf{NOTE.} Suppose that $t_i = (a_i, b_i)$. Let $G$ be the graph on $[n]$ with edges $a_i, b_i$, $1 \leq i \leq k$. Then the statement that the $t_i$’s generate $\mathfrak{S}_n$ is equivalent to the statement that $G$ is connected.