Problem 1.1

A random variable \( X \) has \( \chi^2_n \) (chi-squared with \( n \) degrees of freedom) if it has the same distribution as \( Z_1^2 + \ldots + Z_n^2 \), where \( Z_1, \ldots, Z_n \) are iid \( \mathcal{N}(0, 1) \).

(a) Let \( Z \sim \mathcal{N}(0, 1) \). Show that the moment generating function of \( Y = Z^2 - 1 \) satisfies
\[
\phi(s) := E[e^{sY}] = \begin{cases} 
\frac{e^{-s}}{\sqrt{1 - 2s}} & \text{if } s < 1/2 \\
\infty & \text{otherwise}
\end{cases}
\]

(b) Show that for all \( 0 < s < 1/2 \),
\[
\phi(s) \leq \exp \left( \frac{s^2}{1 - 2s} \right).
\]

(c) Conclude that
\[
\mathbb{P}(Y > 2t + 2\sqrt{t}) \leq e^{-t}
\]
[Hint: you can use the convexity inequality \( \sqrt{1 + u} \leq 1 + u/2 \)].

(d) Show that if \( X \sim \chi^2_n \), then, with probability at least \( 1 - \delta \), it holds
\[
X \leq n + 2\sqrt{n \log(1/\delta)} + 2 \log(1/\delta).
\]

Problem 1.2

Let \( A = \{A_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq m} \) be a random matrix such that its entries are iid sub-Gaussian random variables with variance proxy \( \sigma^2 \).

(a) Show that the matrix \( A \) is sub-Gaussian. What is its variance proxy?

(b) Let \( \|A\| \) denote the operator norm of \( A \) defined by
\[
\max_{x \in \mathbb{R}^m} \frac{|Ax|_2}{|x|_2}.
\]
Show that there exits a constant \( C > 0 \) such that
\[
\mathbb{E}\|A\| \leq C(\sqrt{m} + \sqrt{n}).
\]
Problem 1.3

Let \( K \) be a compact subset of the unit sphere of \( \mathbb{R}^p \) that admits an \( \varepsilon \)-net \( N_\varepsilon \) with respect to the Euclidean distance of \( \mathbb{R}^p \) that satisfies \( |N_\varepsilon| \leq (C/\varepsilon)^d \) for all \( \varepsilon \in (0, 1) \). Here \( C \geq 1 \) and \( d \leq p \) are positive constants. Let \( X \sim \text{subG}_p(\sigma^2) \) be a centered random vector.

Show that there exists positive constants \( c_1 \) and \( c_2 \) to be made explicit such that for any \( \delta \in (0, 1) \), it holds

\[
\max_{\theta \in K} \theta^\top X \leq c_1 \sigma \sqrt{d \log(2p/d)} + c_2 \sigma \sqrt{\log(1/\delta)}
\]

with probability at least \( 1 - \delta \). Comment on the result in light of Theorem 1.19.

Problem 1.4

Let \( X_1, \ldots, X_n \) be \( n \) independent and random variables such that \( \mathbb{E}[X_i] = \mu \) and \( \text{var}(X_i) \leq \sigma^2 \). Fix \( \delta \in (0, 1) \) and assume without loss of generality that \( n \) can be factored into \( n = K \cdot G \) where \( G = 8 \log(1/\delta) \) is a positive integer.

For \( g = 1, \ldots, G \), let \( \bar{X}_g \) denote the average over the \( g \)th group of \( k \) variables. Formally

\[
\bar{X}_g = \frac{1}{k} \sum_{i=(g-1)k+1}^{gk} X_i.
\]

1. Show that for any \( g = 1, \ldots, G \),

\[
\mathbb{P}[\bar{X}_g - \mu > \frac{2\sigma}{\sqrt{k}}] \leq \frac{1}{4}.
\]

2. Let \( \hat{\mu} \) be defined as the median of \( \{\bar{X}_1, \ldots, \bar{X}_G\} \). Show that

\[
\mathbb{P}[\hat{\mu} - \mu > \frac{2\sigma}{\sqrt{K}}] \leq \mathbb{P}[B \geq \frac{G}{2}],
\]

where \( B \sim \text{Bin}(G, 1/4) \).

3. Conclude that

\[
\mathbb{P}[\hat{\mu} - \mu > 4\sigma \sqrt{\frac{2\log(1/\delta)}{n}}] \leq \delta
\]

4. Compare this result with Corollary 1.7 and Lemma 1.4. Can you conclude that \( \hat{\mu} - \mu \sim \text{subG}(\bar{\sigma}^2/n) \) for some \( \bar{\sigma}^2 \)? Conclude.
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