Yielding Under Multi-axial Stress

2.002 Mechanics and Materials II
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Uniaxial tension/compression:
  initial linear elastic response,
as axial stress, $\sigma$, is increased
up to the uniaxial “yield condition”:

$$\left| \sigma \right| \leq \sigma_y$$

Suppose that, at some location
in a body made of the same material,
the state of stress is multi-axial, with
cartesian components $\sigma_{ij}$;

**QUESTION:** Will plastic deformation
occur under
this state of stress?
Approach: we need to define a non-negative scalar, stress-valued function of [all] the stress components, such that it can consistently generalize the uniaxial yield criterion, $|\sigma| < \sigma_y$

Observation # 1: pressure insensitivity of uniaxial yielding

Suppose that a uniaxial test is performed under fixed superposed hydrostatic pressure, $p$, so the cartesian stress components are

$$
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}
= 
\begin{bmatrix}
\sigma - p & 0 & 0 \\
0 & -p & 0 \\
0 & 0 & -p
\end{bmatrix}
$$

Plastic deformation is observed to commence when $|\sigma| = \sigma_y$, essentially independent of the value of $p$

*This suggests that yielding is ~ independent of the mean normal stress given by $\Sigma = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$*
Recall the stress deviator tensor, whose components are given by

\[
\begin{bmatrix}
\sigma_{ij}^{(\text{dev})}
\end{bmatrix} \equiv \begin{bmatrix}
\sigma_{ij}
\end{bmatrix} - \frac{1}{3} \left( \sum_{k=1}^{3} \sigma_{kk} \right) \begin{bmatrix}
\delta_{ij}
\end{bmatrix}
\]

Clearly, the stress deviator tensor is independent of the mean normal stress.

The Mises equivalent tensile stress is defined, for any state of stress, \( \sigma_{ij} \), in terms of the components of the corresponding stress deviator tensor by

\[
\bar{\sigma} \equiv \sqrt{\frac{3}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij}^{(\text{dev})} \sigma_{ij}^{(\text{dev})}} \geq 0
\]

The yield condition for general multiaxial states of stress can be expressed as

\[
\bar{\sigma} \leq \sigma_y
\]
Is our general criterion for multiaxial yielding consistent with our previously-established uniaxial yield criterion $|\sigma| = \sigma_y$?

**Uniaxial stress:**

$$
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix} =
\begin{bmatrix}
\sigma & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

**Stress deviator:**

$$
\begin{bmatrix}
\sigma_{11}^{(\text{dev})} & \sigma_{12}^{(\text{dev})} & \sigma_{13}^{(\text{dev})} \\
\sigma_{21}^{(\text{dev})} & \sigma_{22}^{(\text{dev})} & \sigma_{23}^{(\text{dev})} \\
\sigma_{31}^{(\text{dev})} & \sigma_{32}^{(\text{dev})} & \sigma_{33}^{(\text{dev})}
\end{bmatrix} =
\begin{bmatrix}
\frac{2\sigma}{3} & 0 & 0 \\
0 & -\frac{\sigma}{3} & 0 \\
0 & 0 & -\frac{\sigma}{3}
\end{bmatrix}
$$

**Mises stress measure:**

$$
\bar{\sigma} = \sqrt{\frac{3}{2} \left\{ \left( \frac{2\sigma}{3} \right)^2 + \left( -\frac{\sigma}{3} \right)^2 + \left( -\frac{\sigma}{3} \right)^2 \right\}}
$$

$$
= |\sigma| \sqrt{\frac{3}{2} \left\{ \frac{4}{9} + \frac{1}{9} + \frac{1}{9} \right\}}
$$

$$
= |\sigma|
$$

Mises yield specializes to the uniaxial yield condition under uniaxial stress:

$$
\bar{\sigma} = \sigma_y \iff |\sigma| = \sigma_y
$$
Equivalent Expressions for Mises Equivalent Tensile Stress

In terms of stress deviator components:
\[ \bar{\sigma} \equiv \sqrt{\frac{3}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij}^{(\text{dev})} \sigma_{ij}^{(\text{dev})}} \geq 0 \]

In terms of stress components:
\[ \bar{\sigma} = \sqrt{\frac{1}{2} \left[ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right] + 3 \left[ \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \right]} \]

In terms of principal stress values:
\[ \bar{\sigma} = \sqrt{\frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]} \]
EXAMPLE: Combined tension and torsion of a thin-walled tube:

Stress components and relation to loads and tube geometry:

\[
\begin{bmatrix}
\sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\
\sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\
\sigma_{z r} & \sigma_{z\theta} & \sigma_{zz}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \sigma_{\theta z} \\
0 & \sigma_{z\theta} & \sigma_{zz}
\end{bmatrix}
\]

\[
\sigma_{zz} \equiv \frac{F}{2\pi R t} \equiv \sigma''; \quad \sigma_{\theta z} \equiv \frac{M_t}{2\pi R^2 t} \equiv \tau''
\]

Stress deviator components:

\[
\begin{bmatrix}
\sigma_{ij}^{(\text{dev})}
\end{bmatrix}
= \begin{bmatrix}
\frac{-\sigma}{3} & 0 & 0 \\
0 & \frac{-\sigma}{3} & \tau \\
0 & \tau & \frac{2\sigma}{3}
\end{bmatrix}
\]

Evaluate Mises stress and compare to Uniaxial yield strength

\[
\bar{\sigma}^2 = \sigma^2 + 3\tau^2 \leq \sigma_y^2
\]

The Mises yield condition for this stress state can be represented as an ellipse in a 2D space whose axes are \(\sigma''\) and \(\tau''\)
EXAMPLE (continued)

A tube of wall thickness $t = 3$ mm and mean radius $\bar{R} = 30$ mm is made of a material having tensile yield strength $\sigma_y = 500$ MPa and is preloaded to an axial force $F = 200$ kN.

What is the maximum torque that can be applied without causing yield in the tube?

Rearrange Mises yield:

$$3\tau^2 \leq \sigma_y^2 - \sigma^2$$

Load/stress/geometry:

$$3 \left( \frac{M_t}{2\pi \bar{R}^2 t} \right)^2 \leq \sigma_y^2 - \left( \frac{F}{2\pi \bar{R} t} \right)^2$$

Algebra...

$$|M_t| \leq \frac{2\pi \bar{R}^2 t}{\sqrt{3}} \sigma_y \sqrt{1 - \left( \frac{F}{2\pi \bar{R} t \sigma_y} \right)^2}$$

Numerical values & units

$$|M_t| \leq \frac{2\pi (30\text{mm})^2 \times 3\text{mm}}{\sqrt{3}} \frac{500\text{N}}{\text{mm}^2} \sqrt{1 - \left( \frac{2 \times 10^5\text{N}}{2\pi 30\text{mm} \times 3\text{mm} \times \frac{500\text{N}}{\text{mm}^2}} \right)^2}$$

$$\leq 3.46 \text{ kNm}$$

ANSWER: