Creep and Creep Fracture: Part II
Stress and Deformation Analysis in Creeping Structures
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Steady-State Bending of Viscoplastic Beams

1. Kinematics

\[ v(x, t) \] 
\[ \kappa(x, t) \triangleq \frac{\partial^2 v(x,t)}{\partial x^2} \] 
\[ \epsilon(x, y, t) = -\kappa(x, t)y \] 
\[ \dot{\epsilon}(x, y, t) - \kappa(x, t)y \] 
\[ \dot{\kappa}(x, y, t) \triangleq \frac{\partial^2 \dot{v}(x,t)}{\partial x^2} \]

\[ |\dot{\epsilon}| \text{sgn}(\dot{\epsilon}) = -|\dot{\kappa}| \text{sgn}(\dot{\kappa})|y| \text{sgn}(y) \]
\[ = |\dot{\kappa}| |y| [- \text{sgn}(\dot{\kappa}) \text{sgn}(y)] \]

Therefore, \[ |\dot{\epsilon}| = |\dot{\kappa}| |y| \] and
\[ \text{sgn}(\dot{\epsilon}) = -\text{sgn}(\dot{\kappa}) \text{sgn}(y) \]
2. Constitutive Relation

\[ \dot{\varepsilon} = \dot{\varepsilon}^c = \dot{\varepsilon}_0 \left( \frac{|\sigma|}{s} \right)^n \text{sgn}(\sigma) \]

\[ \sigma = s \left( \frac{|\dot{\varepsilon}|}{\dot{\varepsilon}_0} \right)^{1/n} \text{sgn}(\dot{\varepsilon}) \]

\[ \Rightarrow \sigma(x, y, t) = s \left( \frac{|\dot{\kappa}(x, t)||y|}{\dot{\varepsilon}_0} \right)^{1/n} \text{sgn}(\dot{\varepsilon}) \]

\[ \Rightarrow \sigma(x, y, t) = \{-\text{sgn}(\dot{\kappa})\text{sgn}(y)s\} \left( \frac{|\dot{\kappa}(x, t)|}{\dot{\varepsilon}_0} \right)^{1/n} |y|^{1/n} \]

\[ \dot{\varepsilon}_0 \quad \text{reference strain rate} \]

\[ s > 0 \quad \text{reference stress} \]
3. Moment-Curvature Rate Relation

\[ M(x, t) = -\int_A y\sigma(x, y, t) dA \]

\[ = \{\text{sgn}(\dot{\kappa})s\} \left(\frac{|\dot{\kappa}(x, t)|}{\dot{\epsilon}_0}\right)^{1/n} \int_A \text{sgn}(y)|y|^{1+1/n} dA \]

with \( I_n \equiv \int_A \text{sgn}(y)|y|^{1+1/n} dA \)

\[ M(x, t) = \{\text{sgn}(\dot{\kappa})s\} \left(\frac{|\dot{\kappa}(x, t)|}{\dot{\epsilon}_0}\right)^{1/n} I_n \]
4. Equation for Stress

\[
\sigma(x, y, t) = - \{\text{sgn}(\dot{\kappa})s\} \left( \frac{\dot{\kappa}(x, t)}{\dot{\epsilon}_0} \right)^{1/n} |y|^{1/n}(\text{sgn}(y))
\]

\[
\Rightarrow \sigma(x, y, t) = - \frac{M(x, t)}{I_n} |y|^{1/n} \text{sgn}(y)
\]

5. Differential Equation for Lateral Displacement

\[
|M(x, t)| = \left( \frac{\dot{\kappa}(x, t)}{\dot{\epsilon}_0} \right)^{1/n} sI_n
\]

\[
\Rightarrow \dot{\kappa}(x, t) = \dot{\epsilon}_0 \left\{ \frac{|M(x, t)|}{sI_n} \right\}^n \text{sgn}(M(x, t))
\]

\[
\frac{\partial^2 \dot{v}(x, t)}{\partial x^2} = \dot{\epsilon}_0 \left\{ \frac{|M(x, t)|}{sI_n} \right\}^n \text{sgn}(M(x, t))
\]
Example Problem: Cantilever Beam

\[
M(x, t) - P(t)(L - x) = 0
\]
\[
M(x, t) = P(t)(L - x) \quad 0 \leq x \leq L
\]
\[
\dot{\kappa}(x) = \frac{\partial^2 \dot{v}}{\partial x^2} = \dot{\varepsilon}_0 \left\{ \frac{|M|}{sI_n} \right\}^n \text{sgn}(M)
\]
\[
= \dot{\varepsilon}_0 \left\{ \frac{|P(t)(L - x)|}{sI_n} \right\}^n
\]
\[
= \dot{\varepsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n (L - x)^n
\]
Example Problem: Cantilever Beam (cont.)

Boundary conditions: (1) $\dot{v} = 0$ at $x = 0$ and (2) $\frac{\partial \dot{v}}{\partial x} = 0$ at $x = 0$ (Assume $P(t) > 0$)

\[
\frac{\partial \dot{v}}{\partial x} = \dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n \left\{ -\frac{1}{n+1} \right\} (L - x)^{n+1} + C_1
\]

\[
C_1 = \dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n \frac{L^{n+1}}{n+1} \quad \text{(Using BC (2))}
\]

\[
\frac{\partial \dot{v}}{\partial x} = \dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n \frac{1}{n+1} \left[ -(L - x)^{n+1} + L^{n+1} \right]
\]

\[
\dot{v} = \dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n \frac{1}{n+1} \left[ \frac{(L - x)^{n+2}}{n+2} + L^{n+1}x \right] + C_2
\]

\[
C_2 = -\dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n \frac{1}{n+1} \left[ \frac{L^{n+2}}{n+2} \right] \quad \text{(Using BC (1))}
\]
Example Problem: Cantilever Beam (cont.)

\[ \dot{v} = \dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n \frac{1}{n + 1} \left[ \frac{(L - x)^{n+2}}{n + 2} + L^{n+1}x - \frac{L^{n+2}}{n + 2} \right] \]

\[ \dot{\delta} = |\dot{v}(x = L)| = \dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n L^{n+2} \frac{n}{n + 2} \text{ (Tip deflection rate)} \]

\[ |P(t)| = \left[ \frac{(\dot{\delta}/\dot{\epsilon}_0)(n + 2)}{L^{n+2}} \right]^{1/n} sI_n \]
Three-Dimensional Generalization of Constitutive Equations for Elastic-Viscoplastic Materials

1. Strain Rate Decomposition:

\[ \dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^c \]

- \( \dot{\varepsilon}_{ij} \) total strain rate
- \( \dot{\varepsilon}_{ij}^e \) elastic strain rate
- \( \dot{\varepsilon}_{ij}^c \) creep or viscoplastic strain rate

2. Constitutive Equations for \( \dot{\varepsilon}_{ij}^e \):

\[ \dot{\varepsilon}_{ij}^e = \frac{1}{E} \left[ (1 + \nu)\dot{\sigma}_{ij} - \nu \left( \sum_k \dot{\sigma}_{kk} \right) \delta_{ij} \right] \]
Young’s modulus
\\nu \quad \text{Poisson’s ratio}

2. Constitutive Equations for $\dot{\epsilon}_c^{ij}$:

\[
\dot{\epsilon}_c^{ij} = \dot{\epsilon}^c (3/2) \left\{ \sigma'_ij/\sigma \right\} \quad \text{creep strain rate components}
\]
\[
\dot{\epsilon}^c = \dot{\epsilon}_0 \left\{ \bar{\sigma}/s \right\}^n \quad \text{equivalent tensile creep rate}
\]
\[
\sigma'_ij = \sigma_{ij} - (1/3) \left( \sum_k \sigma_{kk} \right) \delta_{ij} \quad \text{stress deviator components}
\]
\[
\bar{\sigma} = \sqrt{(3/2) \sum_{i,j} \sigma'_ij \sigma'_ij} \quad \text{Mises equivalent tensile stress}
\]
\[
= \left\{ (1/2) \left\{ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right\} + 3 \left\{ \sigma^2_{12} + \sigma^2_{23} + \sigma^2_{31} \right\} \right\}^{1/2}
\]
\( n \) creep exponent
\( \dot{\varepsilon}_0 \) reference strain rate
\( s \) reference stress

- Note that in uniaxial tension when \( \sigma_{11} = \sigma \), with all other \( \sigma_{ij} = 0 \), we have \( \sigma'_{11} = (2/3)\sigma \), \( \sigma'_{22} = \sigma'_{33} = -(1/3)\sigma \), and \( \bar{\sigma} = |\sigma| \). Therefore, the constitutive equation for \( \dot{\varepsilon}_{ij}^c \) yields

\[
\dot{\varepsilon}_{11}^c = \dot{\varepsilon}_0 \left\{ \bar{\sigma}/s \right\}^n \text{sgn}(\sigma)
\]

\[
\dot{\varepsilon}_{22}^c = \dot{\varepsilon}_{33}^c = -(1/2)\dot{\varepsilon}_{11}^c
\]

\[
\dot{\varepsilon}_{i,j}^c = 0 \text{ otherwise,}
\]
as it should.

• For the case of rigid-viscoplastic materials, the elastic strains and strain rates are neglected:

$$\varepsilon_{ij} = \varepsilon_{ij}^c$$

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^c = \dot{\varepsilon}_c^c (3/2) \{\sigma'_{ij}/\sigma\}$$

$$\ddot{\varepsilon}_c = \ddot{\varepsilon}_0 \{\bar{\sigma}/s\}^n$$
Example Problem: Torsion of Thin-walled Tube

- Consider a thin-walled tube of radius \( r \), wall thickness \( t \) and length \( L \). One end of the tube is fixed, while on the other a constant twisting moment \( M_t \) is applied. The tube is at high homologous temperatures (creep conditions prevail). Calculate the twisting rate \( \dot{\phi} \) for the tube.
Example Problem: [Thin-Walled] Torsion

- The angle of twist $\phi$ is a function of time, i.e., $\phi = \phi(t)$. The angle of twist per unit length is denoted by $\alpha = \phi/L = \alpha(t)$

- Displacement field:

$$u_r = 0$$

$$u_\theta = \alpha z r$$

$$u_z = 0$$
• Strain field:

\[
\epsilon_{\theta z} = \frac{1}{2} \left[ \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right] = \frac{1}{2} \alpha r, \\
\text{with all other} \quad \epsilon_{ij} = 0
\]

• Therefore the strain rate is

\[
\begin{align*}
\dot{\epsilon}_{\theta z} &= \frac{1}{2} \dot{\alpha} r \\
\dot{\epsilon}_{\theta z} &= \frac{1}{2} \frac{\phi}{2 L} r 
\end{align*}
\] (1)
Constitutive equation: Since the applied moment \( M(t) \) is constant, the elastic strain rate \( \dot{\epsilon}_{ij}^e = 0 \). (Strictly, we need to verify that \( \dot{\sigma}_{ij} = 0 \); however, the thin-walled tube in torsion has one constant non-zero stress component, \( \sigma_{\theta z} \), that is directly proportional to twisting moment, \( M_t \) (see ‘Equilibrium’, below)). Therefore,

\[
\begin{align*}
\dot{\epsilon}_{ij} &= \dot{\epsilon}_c \frac{3}{2} \left\{ \frac{\sigma'_{ij}}{\bar{\sigma}} \right\} \\
\Rightarrow \dot{\epsilon}_\theta z &= \dot{\epsilon}_c \frac{3}{2} \left\{ \frac{\sigma'_{\theta z}}{\bar{\sigma}} \right\} \\
\dot{\epsilon}_c &= \dot{\epsilon}_0 \left\{ \frac{\bar{\sigma}}{s} \right\}^n \\
\sigma'_{ij} &= \sigma_{ij} - \frac{1}{3} \left( \sum_k \sigma_{kk} \right) \delta_{ij}
\end{align*}
\]
• The only non-zero stress component is $\sigma_{\theta z}' = \sigma_{\theta z}$. Also, $\bar{\sigma} = \sqrt{3}|\sigma_{\theta z}|$ from the definition of the equivalent tensile stress. Therefore,

$$
\dot{\epsilon}_{\theta z} = \frac{\sqrt{3}}{2} \dot{\epsilon}_0 \left\{ \frac{\sqrt{3}|\sigma_{\theta z}|}{s} \right\}^n \text{sgn}(\sigma_{\theta z}) \quad (2)
$$

• Equilibrium: The applied torque should balance the internal torque of the only non-zero stress component, $\sigma_{\theta z}$.
\[ dM_t = r dF = r \sigma_{\theta z} \, dA = r \sigma_{\theta z} (r d\theta t) = \sigma_{\theta z} t r^2 \, d\theta \]

\[ M_t = \int_0^{2\pi} \sigma_{\theta z} t r^2 \, d\theta = 2\pi t r^2 \sigma_{\theta z} \]

\[ \Rightarrow \sigma_{\theta z} = \frac{M_t}{2\pi t r^2} \quad (3) \]
\[ \dot{\epsilon}_{\theta z} = \frac{\sqrt{3}}{2} \dot{\epsilon}_0 \left\{ \frac{\sqrt{3} |\sigma_{\theta z}|}{s} \right\}^n \text{sgn}(\sigma_{\theta z}) \]

\[ \Rightarrow \frac{1}{2L} \dot{r} = \frac{\sqrt{3}}{2} \dot{\epsilon}_0 \left\{ \frac{\sqrt{3} |M_t|}{s} \frac{1}{2\pi tr^2} \right\}^n \text{sgn}(M_t) \]

\[ \Rightarrow \dot{\phi} = \sqrt{3} \dot{\epsilon}_0 \left[ \frac{\sqrt{3} |M_t|}{2\pi tr^2 s} \right]^n \left( \frac{L}{r} \right) \text{sgn}(M_t) \]

\[ \Rightarrow \dot{\phi} = \sqrt{3} \dot{\epsilon}_0 \left[ \frac{\sqrt{3} M_t}{2\pi tr^2 s} \right]^n \left( \frac{L}{r} \right) \quad \text{for } M_t > 0 \]
Example Problem: Torsion of Thick-walled Tube

\[ \frac{1}{2} \dot{\alpha} r = \dot{\epsilon}_{\theta z}(r) = \frac{\sqrt{3}}{2} \dot{\epsilon}_0 \left( \frac{\sqrt{3} |\sigma_{\theta z}(r)|}{s} \right)^n \text{sgn}(\sigma_{\theta z}(r)) \]

(for \( a \leq r \leq b \))
For simplicity, let $\sigma_{\theta z}(r) > 0$. Therefore,

$$
\dot{\sigma}_{\theta z} = \frac{\sqrt{3}}{2} \dot{\epsilon}_0 \left\{ \frac{\sqrt{3} |\sigma_{\theta z}|}{s} \right\}^n \Rightarrow \sigma_{\theta z}(r) = \frac{s}{\sqrt{3}} \left\{ \frac{2}{\sqrt{3} \dot{\epsilon}_0} \dot{\sigma}_{\theta z}(r) \right\}^{1/n}
$$

$$
\dot{\epsilon}_{z \theta}(r) = \acute{\alpha} r / 2 \Rightarrow \sigma_{z \theta}(r) \equiv Ar^{1/n},
$$

where

$$
A \equiv \frac{s}{\sqrt{3}} \left\{ \frac{\dot{\phi}}{\sqrt{3} \dot{\epsilon}_0 L} \right\}^{1/n}
$$

With $dM_t = \sigma_{\theta z}(r) r^2 dr d\theta = Ar^{2+1/n} dr d\theta$,

$$
M_t = A \int_a^b \int_0^{2\pi} r^{2+1/n} d\theta dr
$$

Let

$$
J_n = \int_a^b \int_0^{2\pi} r^{2+1/n} d\theta dr = \frac{2\pi}{3 + 1/n} \left[ b^{3+1/n} - a^{3+1/n} \right]
$$
Since $A = \frac{M_t}{J_n}$,

$$\sigma_{\theta z}(r) = \frac{M_t}{J_n} r^{1/n}$$

and

$$\frac{s}{\sqrt{3}} \left\{ \frac{\dot{\phi}}{\sqrt{3} \dot{\epsilon}_0 L} \right\}^{1/n} = \frac{M_t}{J_n}$$

$$\Rightarrow \dot{\phi} = \sqrt{3} \dot{\epsilon}_0 \left[ \frac{\sqrt{3} M_t}{s J_n} \right]^n L$$