Euler-Bernoulli Beams: Bending, Buckling, and Vibration

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Linear Elastic Beam Theory

• Basics of beams
  – Geometry of deformation
  – Equilibrium of “slices”
  – Constitutive equations

• Applications:
  – Cantilever beam deflection
  – Buckling of beams under axial compression
  – Vibration of beams
Beam Theory:  
Slice Equilibrium Relations

- $q(x)$: distributed load/length
- $N(x)$: axial force
- $V(x)$: shear force
- $M(x)$: bending moment

**Axial force balance:**

$$0 = N(x + dx) - N(x) \Rightarrow N(x) = \text{constant}$$

**Transverse force balance:**

$$0 = q(x)dx + V(x + dx) - V(x)$$
$$= q(x)dx + \left( V(x) + V'(x)dx + o(dx) \right) - V(x)$$
$$= dx \left[ V'(x) + q(x) \right] \Rightarrow$$
$$0 = V'(x) + q(x) \quad \text{CDL(3.11)}$$

**Moment balance about ‘x+dx’:**

$$0 = V(x)dx + M(x + dx) - M(x) - (q(x)dx)dx$$
$$= V(x)dx + \left( M(x) + M'(x)dx \right) - M(x) - (q(x)dx)dx/2$$
$$= dx \left[ M'(x) + V(x) - q(x)dx/2 \right] \Rightarrow$$
$$0 = M'(x) + V(x) \quad \text{CDL(3.12)}$$
Euler-Bernoulli Beam Theory: Displacement, strain, and stress distributions

Beam theory assumptions on spatial variation of displacement components:

\[
 u(x, y, z) = u_0(x) - yv'(x) \\
 v(x, y, z) = v(x) \\
 w(x, y, z) = 0
\]

Axial strain distribution in beam:

\[
 \epsilon_{xx}(x, y, z) \equiv \frac{\partial u(x, y, z)}{\partial x} = u_0'(x) - yv''(x) \\
 \equiv \epsilon_0(x) - y\kappa(x)
\]

1-D stress/strain relation:

\[
 \sigma_{xx} = E\epsilon_{xx}
\]

Stress distribution in terms of Displacement field:

\[
 \sigma_{xx}(x, y, z) = E \left( \epsilon_0(x) - y\kappa(x) \right)
\]

Axial strain varies linearly Through-thickness at section ‘x’
Slice Equilibrium: Section Axial Force \( N(x) \) and Bending Moment \( M(x) \) in terms of Displacement fields

\( \mathbf{N}(x) \): \( x \)-component of force equilibrium on slice at location ‘\( x \)’:

\[
N(x) \equiv \int \sigma_{xx}(x, y, z) \, dA(y, z) \\
= \int E \{\varepsilon_0(x) - y\kappa(x)\} \, dA \\
= EA\varepsilon_0(x) - E\kappa(x) \int y \, dA.
\]

\( \mathbf{M}(x) \): \( z \)-component of moment equilibrium on slice at location ‘\( x \)’:

\[
M(x) \equiv \int -y \sigma_{xx}(x, y, z) \, dA(y, z) \\
= \int E \{\varepsilon_0(x) + y^2\kappa(x)\} \, dA \\
= -E\varepsilon_0(x) \int y \, dA + E\kappa(x)I
\]

where \( I \equiv \int y^2 \, dA \) is area moment of inertia of cross section
Centroidal Coordinates

\[ \bar{y} = \frac{1}{A} \int y \, dA \]

**choice:** \( \bar{y} \equiv 0 \Rightarrow \int y \, dA = 0 \)

Simplifications:

\[
N(x) = EA\varepsilon_0(x) = EAu'_0(x) \\
M(x) = EI\kappa(x) = EIV''(x)
\]

**Note:** \( I \) is centroidal area moment of inertia:

\[ I \equiv \int y^2 \, dA \]
Tip-Loaded Cantilever Beam: Equilibrium

Free body diagrams:

• statically determinant: support reactions $R$, $M_0$ from equilibrium alone
• reactions “present” because of $x=0$ geometrical boundary conditions $v(0)=0$; $v'(0)=\phi(0)=0$

• general equilibrium equations (CDL 3.11-12) satisfied

How to determine lateral displacement $v(x)$; especially at tip ($x=L$)?
Exercise: Cantilever Beam Under Self-Weight

- Weight per unit length: \( q_0 \)
- \( q_0 = \rho Ag = \rho bhg \)

Free body diagrams:

Find:
- Reactions: \( R \) and \( M_0 \)
- Shear force: \( V(x) \)
- Bending moment: \( M(x) \)
Tip-Loaded Cantilever: Lateral Deflections

Curvature / moment relations:

\[ v''(x) = \frac{1}{EI} M(x) = \frac{1}{EI} \left(P(L - x)\right) \Rightarrow \]

\[ v'(x) = \frac{P}{EI} \left(Lx - x^2/2 + C_1\right) \Rightarrow \]

\[ v(x) = \frac{P}{EI} \left(Lx^2/2 - x^3/6 + C_1x + C_2\right) \]

Tip deflection and rotation:

\[ \Delta \equiv v(L) = \frac{PL^3}{3EI} \]

\[ \Phi \equiv v'(L) = \frac{PL^2}{2EI} \]

Geometric boundary conditions:

\[ \phi(0) = v'(0) = 0 \Rightarrow C_1 = 0 \]

\[ v(0) = 0 \Rightarrow C_2 = 0 \]

\[ v(x) = \frac{Px^2}{6EI} (3L - x) \]

Stiffness and modulus:

\[ k \equiv \frac{P}{\Delta} = \frac{3EI}{L^3} \]

\[ E = \frac{kL^3}{3I} = \frac{PL^3}{3I\Delta} \]
Tip-Loaded Cantilever: Axial Strain Distribution

strain field (no axial force):

\[ \varepsilon_{xx}(x, y) = -yv''(x) \]

\[ = -\frac{yM(x)}{EI} \]

top/bottom axial strain distribution:

\[ \varepsilon_{xx}^{TOP}(x) = -\frac{6P(L-x)}{bh^2E} \quad (y = h/2) \]

\[ \varepsilon_{xx}^{BOTTOM}(x) = \frac{6P(L-x)}{bh^2E} \quad (y = -h/2) \]

\[ I_{rectangle} = \frac{bh^3}{12} \]

strain-gauged estimate of E:

\[ E = \frac{6P(L-x)}{bh^2\varepsilon_{xx}^{BOTTOM}(x)} = \frac{6P(L-x)}{bh^2\varepsilon_{xx}^{TOP}(x)} \]
Euler Column Buckling: Non-uniqueness of deformed configuration

**Free body diagram** (note: evaluated in deformed configuration):

\[ \sum M_Z = 0 \Rightarrow M(x) + P v(x) = 0 \]

**Moment/curvature:**

\[ M(x) = EI \kappa(x) = EI v''(x) \]

**Ode for buckled shape:**

\[
\begin{align*}
0 &= M(x) + P v(x) \\
&= EI v''(x) + P v(x) \\
0 &= v''(x) + \frac{P}{EI} v(x) \\
&\equiv v''(x) + k^2 v(x)
\end{align*}
\]

Note: linear 2nd order ode; Constant coefficients (but parametric: \( k^2 = P/EI \))
Euler Column Buckling, Cont.

**ode for buckled shape:**

\[ 0 = v''(x) + k^2 v(x) \]

**general solution to ode:**

\[ v(x) = C_1 \sin kx + C_2 \cos kx \]

**boundary conditions:**

\[ v(0) = 0 \Rightarrow C_2 = 0 \]

\[ v(L) = 0 \Rightarrow C_1 \sin kL = 0 \Rightarrow \]

\[ C_1 = 0 \text{ (trivial) or } \sin kL = 0 \]

**buckling-based estimate of } E:}

\[ E_{\text{pinned/pinned}} = \frac{P_{\text{crit}} L^2}{\pi^2 I} \]

**parametric consequences:**

Non-trivial buckled shape only when

\[ \sin kL = 0 \Rightarrow kL = n\pi \]

\[ k^2 = (n\pi/L)^2 \]

\[ P = EI k^2 = \frac{n^2 EI \pi^2}{L^2} \]

\[ P_{\text{crit}}(n = 1) = \frac{\pi^2 EI}{L^2} \]
Euler Column Buckling: General Observations

• buckling load, $P_{\text{crit}}$, is proportional to $EI/L^2$

• proportionality constant depends strongly on boundary conditions at both ends:

  • the more kinematically restrained the ends are, the larger the constant and the higher the critical buckling load (see Lab 1 handout)

• safe design of long slender columns requires adequate margins with respect to buckling

• buckling load may occur at a compressive stress value ($\sigma = P/A$) that is less than yield stress, $\sigma_y$
Euler-Bernoulli Beam Vibration

assume time-dependent lateral motion:
\[ v(x, t) = \bar{v}(x) \sin \omega t \]

lateral velocity of slice at ‘x’:
\[ \frac{\partial v(x, t)}{\partial t} \equiv \dot{v}(x, t) = \omega \bar{v}(x) \cos \omega t \]

lateral acceleration of slice at ‘x’:
\[ \frac{\partial^2 v(x, t)}{\partial t^2} \equiv \ddot{v}(x, t) = -\omega^2 \bar{v}(x) \sin \omega t \]

mass of dx-thickness slice:
\[ dm = dx \rho A \]

linear momentum balance (Newton):
\[ \sum F_y = dm \ddot{v}(x, t) \Rightarrow \]
\[ 0 = dx \left( M''(x, t) - \rho \omega^2 A \bar{v}(x) \sin \omega t \right) \]

net lateral force (q(x,t)=0):
\[ \sum F_y = dx \frac{\partial V(x, t)}{\partial x} \equiv dx V'(x, t) \]

moment balance:
\[ 0 = \frac{\partial M(x, t)}{\partial x} + V(x, t) \equiv M'(x, t) + V(x, t) \Rightarrow \]
\[ 0 = M''(x, t) + V'(x, t) \]
Euler-Bernoulli Beam Vibration, Cont.

linear momentum balance:  
\[ 0 = M''(x, t) - \rho \omega^2 A \ddot{v}(x) \sin \omega t \]

moment/curvature:  
\[ M(x, t) = EI \kappa(x, t) = EI \dddot{v}(x) \sin \omega t \]

ode for mode shape, \( v(x) \), and vibration frequency, \( \omega \):  
\[ 0 = M''(x, t) - \rho \omega^2 A \ddot{v}(x) \sin \omega t \]
\[ = \sin \omega t \left( EI \dddot{v}(x) - \rho \omega^2 A \ddot{v}(x) \right) \]
\[ = \sin \omega t \left( \dddot{v}(x) - \frac{\rho \omega^2 A \ddot{v}(x)}{EI} \right) \]
\[ \equiv \sin \omega t \left( \dddot{v}(x) - \beta^4 \ddot{v}(x) \right) \Rightarrow \]
\[ 0 = \dddot{v}(x) - \beta^4 \ddot{v}(x) \]

general solution to ode:  
\[ \dddot{v}(x) = A_1 \sin \beta x + A_2 \cos \beta x + A_3 \sinh \beta x + A_4 \cosh \beta x \]
Euler-Bernoulli Beam Vibration, Cont(2)

**general solution to ode:**

\[ \ddot{v}(x) = A_1 \sin \beta x + A_2 \cos \beta x + A_3 \sinh \beta x + A_4 \cosh \beta x \]

**pinned/pinned boundary conditions:**

\[ \ddot{v}(0) = 0 \Rightarrow A_2 + A_4 = 0 \]
\[ \ddot{v}''(0) = 0 \Rightarrow \beta^2 (-A_2 + A_4) = 0 \]
\[ \ddot{v}(L) = 0 \Rightarrow A_1 \sin \beta L + A_3 \sinh \beta L = 0 \]
\[ \ddot{v}''(L) = 0 \Rightarrow \beta^2 (-A_1 \sin \beta L + A_3 \sinh \beta L) = 0 \]

**pinned/pinned restricted solution:**

\[ \beta \neq 0; \quad A_2 = A_3 = A_4 = 0; \]

\[ A_1 \sin \beta L = 0 \Rightarrow \]

\[ A_1 = 0 \text{ (trivial), OR} \sin \beta L = 0 \Rightarrow \beta L = n\pi \]

**Solution (n=1, first mode):**

\[ A_1: \text{‘arbitrary’ (but small)} \]
\[ \text{vibration amplitude} \]

\[ \beta_1 = \pi / L \Rightarrow \]

\[ \ddot{v}(n=1) (x, t) = A_1 \sin(\pi x / L) \sin \omega_1 t \]

\[ \beta_1^4 = (\pi / L)^4 = \omega_1^2 \rho A / EI \Rightarrow \]

\[ \omega_1 = \sqrt{\frac{EI \pi^4}{\rho AL^4}} \]

**\( \tau_1: \text{period of first mode} \)**

\[ \tau_1 = \frac{2\pi}{\omega_1} \]