2.002 MECHANICS & MATERIALS II

STRESS-STRAIN RELATIONS

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Introduction To Linear Elasticity

- Elastic materials have a reference shape to which they will return if the forces applied to them are removed (provided the forces are not too large).

- An elastic material is one in which the stress arises in response to the change in shape, that is the strain $\varepsilon$, that the body has undergone from its reference configuration.

- The stress is independent of the past history of strain, as well as the rate at which the strain is changing with time.
• The behavior of an **elastic material** under isothermal conditions is described by a constitutive equation of the form

\[ \sigma_{ij} = \sigma_{ij}(\varepsilon_{kl}), \quad \sigma = \sigma(\varepsilon). \]

• For a **linear elastic material** the symmetric tensor valued function \( \sigma(\varepsilon) \) is linear in its argument:

\[ \sigma_{ij}(\varepsilon_{kl}) = \sum_{k,l} C_{ijkl} \varepsilon_{kl}, \quad \sigma(\varepsilon) = \mathbf{C} [\varepsilon]. \]

The stress components are linear functions of the infinitesimal strain components. The fourth-order tensor \( \mathbf{C} \) which linearly maps the second order tensor \( \varepsilon \) to the second order tensor \( \sigma \), is called the **stiffness tensor**. The \( 3^4 = 81 \) constants \( C_{ijkl} \) are called the elastic moduli.
• An elastic material does not dissipate energy.

• There exist a scalar valued function of the strain $\epsilon$, 

$$W(\epsilon)$$

called the strain energy density per unit reference volume, such that the stress $\sigma_{ij}$ is the derivative of $W(\epsilon)$; that is

$$\sigma_{ij} = \frac{\partial W(\epsilon)}{\partial \epsilon_{ij}}.$$
Expanding the strain energy function \( W(\epsilon) \) about the undeformed state, \( \epsilon = 0 \), we have

\[
W(\epsilon) = W(0) + \sum_{i,j} \frac{\partial W(0)}{\partial \epsilon_{ij}} \epsilon_{ij} + \frac{1}{2} \sum_{i,j,k,l} \frac{\partial^2 W(0)}{\partial \epsilon_{ij} \partial \epsilon_{kl}} \epsilon_{ij} \epsilon_{kl} + \cdots
\]

\( W(0) = 0 \)  No strain energy at zero strain

\( \sigma_{ij,\text{residual}} = \frac{\partial W(0)}{\partial \epsilon_{ij}} \)  Residual stress at zero strain, neglect

\( C_{ijkl} = \frac{\partial^2 W(0)}{\partial \epsilon_{ij} \partial \epsilon_{kl}} \)  Elastic moduli – constants; \( 3^4 = 81! \)

Thus, for a \textbf{linear elastic material}, the strain energy density function \( W \) is quadratic in \( \epsilon \):

\[
W(\epsilon) = \frac{1}{2} \sum_{i,j,k,l} C_{ijkl} \epsilon_{ij} \epsilon_{kl}.
\]
\[ W(\varepsilon) = \frac{1}{2} \sum_{i,j,k,l} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \]

\[
\frac{\partial^2 W(\mathbf{0})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = \frac{\partial^2 W(\mathbf{0})}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}} \implies C_{ijkl} = C_{klij} \\
\varepsilon_{ij} = \varepsilon_{ji} \implies C_{ijkl} = C_{jikl} \\
\varepsilon_{kl} = \varepsilon_{lk} \implies C_{ijkl} = C_{ijlk}
\]

these symmetries reduce the number of independent elastic constants to 21.
Now,

\[ \sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = \frac{1}{2} \frac{\partial}{\partial \varepsilon_{ij}} \left\{ \sum_{p,q,r,s} C_{pqrs} \varepsilon_{pq} \varepsilon_{rs} \right\}, \]

\[ = \frac{1}{2} \sum_{p,q,r,s} C_{pqrs} \left( \delta_{ip} \delta_{jq} \varepsilon_{rs} + \delta_{ir} \delta_{js} \varepsilon_{pq} \right), \]

\[ = \frac{1}{2} \sum_{r,s} \left( C_{ijsr} \varepsilon_{rs} + C_{pqij} \varepsilon_{pq} \right) = \sum_{r,s} C_{ijrs} \varepsilon_{rs}. \]

That is,

\[ \sigma_{ij} = \sum_{k,l} C_{ijkl} \varepsilon_{kl}, \]

which, as noted before, is the constitutive equation for a linear elastic solid. The stress components are linear functions of the infinitesimal strain components.
Constitutive equation for a linear elastic solid:

\[ \sigma_{ij} = \sum_{k,l} C_{ijkl} \varepsilon_{kl}, \quad \sigma = C [\varepsilon], \]

The elastic moduli \( C_{ijkl} \) possess the symmetries

\[ C_{ijkl} = C_{klij}, \]
\[ C_{ijkl} = C_{jikl}, \]
\[ C_{ijkl} = C_{ijlk}. \]

For the most general linear elastic material there are 21 independent elastic moduli \( C_{ijkl} \).
The **stiffness tensor** $C$ is said to be **positive definite** if

$$\sum_{i,j,k,l} C_{ijkl} \epsilon_{ij} \epsilon_{kl} > 0, \text{ for all } \epsilon \neq 0$$

We assume that the $C$ is positive definite. Physically, we assume that the strain energy density $W$ is positive valued, whenever the strain is non-zero. In this case the stress-strain relation

$$\sigma_{ij} = \sum_{k,l} C_{ijkl} \epsilon_{kl}$$

is invertible, such that

$$\epsilon_{ij} = \sum_{k,l} S_{ijkl} \sigma_{kl}.$$ 

The fourth order tensor $S$ is called the **compliance tensor**.
MATERIAL SYMMETRY

Most solids exhibit symmetry properties with respect to certain rotations of the body, or reflection about one or more planes. The effects of these symmetries is to reduce the number of elastic constants from the number 21 for the most general anisotropic material.
Recall that if $\sigma'_{ij} = e'_i \cdot \sigma e'_j$ and $\sigma'_{ij} = e'_i \cdot \sigma e'_j$ are the components of the stress tensor $\sigma$ with respect to the two bases $\{e_i\}$ and $\{e'_i\}$, then $\sigma'_{ij}$ and $\sigma_{ij}$ are related by

$$\sigma'_{ij} = \sum_{k,l} Q_{ik} Q_{jl} \sigma_{kl}.$$ 

This is the tensor transformation law for 2nd-order tensors.

If $S'_{ijkl}$ and $S_{ijkl}$ are the components of the compliance tensor $\mathbf{S}$ with respect to the two bases, then they are related by the tensor transformation law for 4th-order tensors

$$S'_{ijkl} = \sum_{p,q,r,s} Q_{ip} Q_{jq} Q_{kr} Q_{ls} S_{pqrs}.$$ 

Note that these transformation laws hold for all orthogonal matrices $[Q]$. 
If for some $[Q]$ the values of the compliance coefficients in the primed system are the same as those in the unprimed system,

$$S'_{ijkl} = S_{ijkl},$$

or equivalently,

$$S_{ijkl} = \sum_{p,q,r,s} Q_{ip}Q_{jq}Q_{kr}Q_{ls}S_{pqrs},$$

that is if the material properties in the two bases have the same values, then $[Q]$ is called a symmetry transformation.
Isotropy

A material is said to be elastically isotropic if its elastic modulii are invariant with respect to all orthogonal transformations. That is,

\[ S_{ijkl} = \sum_{p,q,r,s} Q_{ip} Q_{jq} Q_{kr} Q_{ls} S_{p,qr,s} \]

holds for all orthogonal matrices \([Q]\).
Working out the details, it may be shown that an isotropic linear elastic material has only two 2 independent elastic constants, and that we may write the strain-stress relation \( \varepsilon_{ij} = \sum_{k,l} S_{ijkl} \sigma_{kl} \) in matrix form as

\[
\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
2\varepsilon_{23} \\
2\varepsilon_{13} \\
2\varepsilon_{12}
\end{pmatrix} =
\begin{pmatrix}
S_{1111} & S_{1122} & S_{1122} & 0 & 0 & 0 \\
S_{1122} & S_{1111} & S_{1122} & 0 & 0 & 0 \\
S_{1122} & S_{1122} & S_{1111} & 0 & 0 & 0 \\
0 & 0 & 0 & \xi & 0 & 0 \\
0 & 0 & 0 & 0 & \xi & 0 \\
0 & 0 & 0 & 0 & 0 & \xi
\end{pmatrix}
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix},
\]

with

\[
\xi = 2(S_{1111} - S_{1122}).
\]
Physical Interpretation of Elastic Moduli For an Isotropic Material

\[
\begin{pmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
2\epsilon_{23} \\
2\epsilon_{13} \\
2\epsilon_{12}
\end{pmatrix}
= 
\begin{pmatrix}
S_{1111} & S_{1122} & S_{1122} & 0 & 0 & 0 \\
S_{1122} & S_{1111} & S_{1122} & 0 & 0 & 0 \\
S_{1122} & S_{1122} & S_{1111} & 0 & 0 & 0 \\
0 & 0 & 0 & \xi & 0 & 0 \\
0 & 0 & 0 & 0 & \xi & 0 \\
0 & 0 & 0 & 0 & 0 & \xi
\end{pmatrix}
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix}, \quad \xi = 2(S_{1111} - S_{1122}).
\]

Consider a uniaxial stress situation \( \sigma_{11} \neq 0 \), all other \( \sigma_{ij} = 0 \). Then

\[
\epsilon_{11} = S_{1111}\sigma_{11}, \quad \epsilon_{22} = \epsilon_{33} = S_{1122}\sigma_{11}.
\]

Defining the Young's Modulus, \( E \), and the Poisson's ratio, \( \nu \), by

\[
E \equiv \frac{\sigma_{11}}{\epsilon_{11}}, \quad \text{and} \quad \nu \equiv -\frac{\epsilon_{22}}{\epsilon_{11}} = -\frac{\epsilon_{33}}{\epsilon_{11}},
\]

we have

\[
E = \frac{1}{S_{1111}}, \quad \text{and} \quad \nu = -\frac{S_{1122}}{S_{1111}}.
\]
Next consider a pure shear stress $\sigma_{12} \neq 0$, all other $\sigma_{ij} = 0$. Then

$$2\varepsilon_{12} = 2(S_{1111} - S_{1122}) \sigma_{12} = \frac{2(1 + \nu)}{E} \sigma_{12}$$

Then, defining the **Shear Modulus**, $G$, for an isotropic material by

$$G \equiv \frac{\sigma_{12}}{2\varepsilon_{12}},$$

we have

$$G \equiv \frac{E}{2(1 + \nu)}$$

Thus

$$\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
2\varepsilon_{23} \\
2\varepsilon_{13} \\
2\varepsilon_{12}
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{pmatrix}
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix}.$$
This strain stress relation may be written in indicial notation as

\[
\varepsilon_{ij} = \left(1 + \nu\right) \sigma_{ij} - \frac{\nu}{E} \left(\sum_k \sigma_{kk}\right) \delta_{ij},
\]

where the two-independent elastic constants are the \textbf{Young's modulus}, \(E\), and the \textbf{Poisson's ratio} \(\nu\).

The shear modulus \(G\) for an isotropic elastic material is given by

\[
G \equiv \frac{E}{2 (1 + \nu)}
\]
Next consider a state of **hydrostatic pressure**

\[ \sigma_{11} = \sigma_{22} = \sigma_{33} = -p, \quad \text{all other} \quad \sigma_{ij} = 0. \]

In this case

\[ \epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \frac{1 - 2\nu}{E} (-p), \quad \text{all other} \quad \epsilon_{ij} = 0, \]

and

\[ \sum_k \epsilon_{kk} = \frac{3(1 - 2\nu)}{E} (-p), \quad \frac{1}{3} \sum_k \sigma_{kk} = -p \]

The **bulk modulus** for an isotropic material is defined by

\[ K \equiv \frac{\frac{1}{3} \sum_k \sigma_{kk}}{\sum_k \epsilon_{kk}} = \frac{\text{mean normal pressure}}{\text{volume change}}. \]

Thus

\[ K \equiv \frac{E}{3(1 - 2\nu)} \]
Considerations of the positive definiteness of the strain energy density $W$ lead to the restrictions

$$E > 0, \quad 0 < \nu < \frac{1}{2}, \quad G > 0, \quad K > 0.$$
Constitutive Relation For Linear Elasticity in terms of $E$ and $\nu$:

\[ \varepsilon_{ij} = \frac{1}{E} \left[ (1 + \nu)\sigma_{ij} - \nu \left( \sum_k \sigma_{kk} \right) \delta_{ij} \right] \]

\[ \sigma_{ij} = \frac{E}{(1 + \nu)} \left[ \varepsilon_{ij} + \frac{\nu}{(1 - 2\nu)} \left( \sum_k \varepsilon_{kk} \right) \delta_{ij} \right] \]
Constitutive Relation For Linear Elasticity in terms of $G$ and $K$:

\[ \sigma'_{ij} = 2G \epsilon'_{ij}, \]

\[ \frac{1}{3} \left( \sum_k \sigma_{kk} \right) = K \left( \sum_k \epsilon_{kk} \right) \]

\[ \epsilon_{ij} = \frac{1}{2G} \left[ \sigma_{ij} - \left( \frac{3K - 2G}{9K} \right) \left( \sum_k \sigma_{kk} \right) \delta_{ij} \right]. \]

\[ \sigma_{ij} = 2G\epsilon_{ij} + \left( K - \frac{2}{3}G \right) \left( \sum_k \epsilon_{kk} \right) \delta_{ij}. \]
## Relations Between Elastic Constants

<table>
<thead>
<tr>
<th></th>
<th>$G$</th>
<th>$K$</th>
<th>$E$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G, \ E$</td>
<td>$\frac{GE}{3(3G-E)}$</td>
<td></td>
<td>$\frac{E-2G}{2G}$</td>
<td></td>
</tr>
<tr>
<td>$G, \ \nu$</td>
<td>$\frac{2G(1+\nu)}{3(1-2\nu)}$</td>
<td></td>
<td>$2G(1+\nu)$</td>
<td></td>
</tr>
<tr>
<td>$G, \ K$</td>
<td></td>
<td>$\frac{9KG}{3K+G}$</td>
<td>$\frac{1}{2} \left[ \frac{3K-2G}{3K+G} \right]$</td>
<td></td>
</tr>
<tr>
<td>$E, \ \nu$</td>
<td>$\frac{E}{2(1+\nu)}$</td>
<td>$\frac{E}{3(1-2\nu)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E, \ K$</td>
<td>$\frac{3EK}{9K-E}$</td>
<td></td>
<td>$\frac{1}{2} \left[ \frac{3K-E}{3K} \right]$</td>
<td></td>
</tr>
<tr>
<td>$\nu, \ K$</td>
<td>$\frac{3K(1-2\nu)}{2(1+\nu)}$</td>
<td></td>
<td>$3K(1-2\nu)$</td>
<td></td>
</tr>
</tbody>
</table>
For metallic materials it is commonly found that

\[ \nu \approx \frac{1}{3}, \quad \implies \quad G \approx \frac{3}{8}E, \quad K \approx E. \]
Thermal strains

In the absence of stress, the strain caused by a small change in temperature from $T_0$ in the reference configuration to $T$ in the current configuration is called the thermal strain. These strains are expressed by the linear relation

$$\varepsilon_{ij}^{\text{thermal}} = A_{ij} (T - T_0)$$

where $A_{ij}$ is called the thermal expansion tensor.

For isotropic materials,

$$A_{ij} = \alpha \delta_{ij},$$

where $\alpha$ is called the coefficient of thermal expansion.
Thermo-Elasticity For Isotropic Materials

\[ \varepsilon_{ij}^{\text{thermal}} = \alpha (T - T_0) \delta_{ij} \]
\[ \varepsilon_{ij}^{\text{mechanical}} = \frac{1}{E} \left[ (1 + \nu)\sigma_{ij} - \nu \left( \sum_k \sigma_{kk} \right) \delta_{ij} \right] \]

For the case of both an application of stress and a change in temperature, the thermo-elastic strains in a linear theory are written as

\[ \varepsilon_{ij} = \varepsilon_{ij}^{\text{mechanical}} + \varepsilon_{ij}^{\text{thermal}}. \]
Hence,

\[
\epsilon_{ij} = \frac{1}{E} \left[ (1 + \nu) \sigma_{ij} - \nu \left( \sum_k \sigma_{kk} \right) \delta_{ij} \right] + \alpha (T - T_0) \delta_{ij},
\]

which can be easily inverted to give

\[
\sigma_{ij} = \frac{E}{(1 + \nu)} \left[ \epsilon_{ij} + \frac{\nu}{(1 - 2\nu)} \left( \sum_k \epsilon_{kk} \right) \delta_{ij} - \frac{(1 + \nu)}{(1 - 2\nu)} \alpha (T - T_0) \delta_{ij} \right].
\]
Failure/Yield Condition

In addition to the small displacement gradient and small temperature change restrictions in the theory of linear elasticity, we need to also explicitly introduce a failure criterion which bounds the levels of stresses beyond which the constitutive equation for isotropic linear elasticity is no longer valid.

For isotropic materials, a simple statement of a failure condition is

$$f(\sigma) \leq \sigma_f,$$

where $f(\sigma)$ is a scalar-valued function of the applied stress $\sigma$, and the scalar number $\sigma_f$ is a material property called the strength of the material.
Isotropy requires that the dependence on $\sigma$ in the function $f(\sigma)$ can only appear in terms of its invariants,

$$f(\text{invariants of } \sigma) \leq \sigma_f.$$ 

or equivalently, in terms of the principal values of the stress,

$$f(\sigma_1, \sigma_2, \sigma_3) \leq \sigma_f.$$

For ductile metallic polycrystalline materials, “failure” of the elastic response occurs when dislocations move large distances through the crystals of a material to produce significant permanent deformation. Thus, for metallic materials the “failure criterion” is actually a “yield criterion.”
It has been found experimentally that for the yielding mechanism in metallic materials the yield function \( f(\text{invariants of } \sigma) \) may be approximated as

\[
f(\text{invariants of } \sigma) = f(\sigma_1, \sigma_2, \sigma_3) \approx \bar{\sigma},
\]

where

\[
\bar{\sigma} = \sqrt{(3/2) \sum_{i,j} \sigma'_{ij} \sigma'_{ij}},
\]

\[
= \left| \frac{1}{2} \left\{ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right\} 
+ 3 \left\{ \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \right\} \right|^{1/2},
\]

\[
= \left| \frac{1}{2} \left\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\} \right|^{1/2},
\]

is called the equivalent tensile stress.
Thus, the \textbf{yield condition} for ductile metallic materials may be written as

\[ \bar{\sigma} \leq \sigma_y, \]

where \( \sigma_y \) is the \textbf{tensile yield strength} of the material.

This yield condition was first proposed by Richard von Mises in 1913, and is known as the Mises yield condition, and \( \bar{\sigma} \) is called the Mises stress.

This yield condition stands for the physical notion that, as long as the equivalent tensile stress \( \bar{\sigma} \) applied on a material is less than the \textbf{material property} \( \sigma_y \), dislocations would not have moved large distances through the crystals of a polycrystalline material to have produced significant permanent deformation.
The strength $\sigma_y$ is typically identified with the 0.2% offset yield strength in a tension (or compression) test and is defined as the stress level from which unloading to zero stress would result in a permanent axial strain of 0.2%.

We will discuss failure/yield conditions for other material classes in a later part of our study on Mechanics of Materials.
Tresca Yield Condition

In terms of the principal stresses, the maximum shear stress in a material at a given point is given by \( \frac{1}{2}(\sigma_1 - \sigma_3) \), and as early as 1864, Henri Edouard Tresca had proposed the yield condition

\[
\frac{1}{2} |\sigma_1 - \sigma_3| \leq \tau_y
\]

for metallic materials, where \( \tau_y \) is the **yield strength in shear**.

In general three-dimensional formulations, there are some mathematical difficulties associated with plasticity theories based on the Tresca yield criterion. It is for this reason that the mathematically more tractable theories of plasticity based on the Mises yield condition are in more wide use these days.
SUMMARY

Limiting ourselves to isothermal situations, we record that the three-dimensional theory of isotropic linear elasticity is based on:

1. The Strain-Displacement Relations

\[ \varepsilon_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right], \quad \varepsilon_{ij} = \varepsilon_{ji}, \quad \left| \frac{\partial u_i}{\partial x_j} \right| \ll 1. \]

2. The Stress-Strain Relations

\[ \sigma_{ij} = \frac{E}{(1 + \nu)} \left[ \varepsilon_{ij} + \frac{\nu}{(1 - 2\nu)} \left( \sum_k \varepsilon_{kk} \right) \delta_{ij} \right], \]
subject to the **yield condition**

\[
\bar{\sigma} \leq \sigma_y,
\]

with

\[
\bar{\sigma} = \sqrt{\frac{3}{2} \sum_{i,j} \sigma'_{ij} \sigma'_{ij}},
\]

\[
= \left[ \frac{1}{2} \left\{ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right\} + 3 \left\{ \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \right\} \right]^{1/2},
\]

\[
= \left[ \frac{1}{2} \left\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\} \right]^{1/2},
\]
3. The Equations of Motion

\[ \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (i = 1, 2, 3). \]

4. Plus Appropriate Boundary Conditions For Surface Tractions and Displacements.
There is a vast amount of engineering, scientific and mathematical literature associated with particular solutions to this set of equations, obtained by using specialized analytical techniques. Since about the end of the 1970's, the digital computer revolution and the associated development of the computational technique called the finite element method have made a major change in how these equations are solved in engineering practice.
• The availability of software incorporating the finite element method and other procedures for solid-modeling and post-processing of results has placed the advanced concepts of elasticity into the hands of a broad community of engineers.

• At the same time, it has created a necessity for them to have a much deeper education and a strong foundation in the underlying physical and mathematical basis of the theory, so that the new computational techniques are used properly to reliably interpret and assess the quality of the approximate solutions they provide.
<table>
<thead>
<tr>
<th>METALS</th>
<th>$E$ (GPa)</th>
<th>$\nu$</th>
<th>$G$ (GPa)</th>
<th>$\alpha$ (10^{-6}/K)</th>
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<tbody>
<tr>
<td>Tungsten</td>
<td>397</td>
<td>0.284</td>
<td>153</td>
<td>4.3 – 4.7</td>
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<td>327</td>
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<td>CERAMICS</td>
<td>( E ) (GPa)</td>
<td>( \nu )</td>
<td>( G ) (GPa)</td>
<td>( \alpha ) (10(^{-6}$/K)</td>
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<td>94 WC, 6 Co</td>
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<tr>
<td>Self-bonded Silicon Carbide</td>
<td>410</td>
<td>0.24</td>
<td>165</td>
<td>4.3</td>
</tr>
<tr>
<td>90 SiC, 10 Si</td>
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<tr>
<td>Sintered Alumina</td>
<td>350</td>
<td>0.23</td>
<td>142</td>
<td>8.5</td>
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<tr>
<td>100 Al(_2)O(_3)</td>
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<tr>
<td>Hot-pressed Silicon Nitride</td>
<td>310</td>
<td>0.25</td>
<td>124</td>
<td>3.2</td>
</tr>
<tr>
<td>96 Si(_3)N(_4), 4MgO</td>
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<tr>
<td>Low-expansion Glass Ceramic</td>
<td>87</td>
<td>0.25</td>
<td>35</td>
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<tr>
<td>2 (Ti, Zr) O(_2), 4 Li(_2)O</td>
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<tr>
<td>20 Al(_2)O(_3), 70 SiO(_2)</td>
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<tr>
<td>Soda-Lime Glass</td>
<td>73</td>
<td>0.21</td>
<td>30</td>
<td>8.5</td>
</tr>
<tr>
<td>13 Na(_2)O, 12(Ca, Mg)O, 72 SiO(_2)</td>
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<tr>
<td>Vitreous Silica 100 SiO(_2)</td>
<td>71</td>
<td>0.17</td>
<td>30</td>
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<tr>
<td>Low-expansion Borosilicate Glass</td>
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<tr>
<td>12 B(_2)O(_3), 4 Na(_2)O, 2 Al(_2)O(_3), 80 SiO(_2)</td>
<td>66</td>
<td>0.2</td>
<td>27.5</td>
<td>4.0</td>
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<tr>
<td>Machineable Glass Ceramic</td>
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<tr>
<td>65 Mica, 35 Glass</td>
<td>64</td>
<td>0.26</td>
<td>25</td>
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<tr>
<td>High-density Molded Graphite</td>
<td>9</td>
<td>0.11</td>
<td>4</td>
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<tr>
<td>POLYMERIC MATERIALS</td>
<td>$E$ (GPa)</td>
<td>$\nu$</td>
<td>$G$ (GPa)</td>
<td>$\alpha$ (10$^{-6}$/K)</td>
</tr>
<tr>
<td>---------------------------------------------</td>
<td>-----------</td>
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<tr>
<td>Polymethylmethacrylate</td>
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<tr>
<td>PMMA</td>
<td>-125°C</td>
<td>6.3</td>
<td>0.26</td>
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<td></td>
<td>25°C</td>
<td>3.7</td>
<td>0.33</td>
<td>1.39</td>
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<td>54-72</td>
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<tr>
<td>Polystyrene</td>
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<tr>
<td>PS</td>
<td>25°C</td>
<td>3.4</td>
<td>0.33</td>
<td>1.28</td>
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<tr>
<td>Polyethylene (low density)</td>
<td>25°C</td>
<td>2.4</td>
<td>0.38</td>
<td>0.87</td>
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<tr>
<td>Polycarbonate</td>
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<td>0.2</td>
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<td>Polyethylene terephthalate</td>
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<td>2</td>
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<td>0.74</td>
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<td>Polyamide (nylon)</td>
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<td>2.8</td>
<td>0.4</td>
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<td>Polyurethane Foam Rubber</td>
<td>25°C</td>
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<td>0.499</td>
<td>0.0005</td>
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<td>Vulcanized Natural Rubber</td>
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<td>VNR</td>
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<tr>
<td>Polyurethane Foam Rubber</td>
<td>25°C</td>
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<td>0.25</td>
<td>0.0002</td>
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<tr>
<td>EUFR</td>
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