Elements of Continuum Elasticity

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Solid Mechanics in 3 Dimensions: stress/equilibrium, strain/displacement, and intro to linear elastic constitutive relations

• Geometry of Deformation
  – Position, 3 components of displacement, and [small] strain tensor
  – Cartesian subscript notation; vectors and tensors
  – Dilatation (volume change) and strain deviator
  – Special cases: homogeneous strain; plane strain

• Equilibrium of forces and moments:
  – Stress and ‘traction’
  – Stress and equilibrium equations
  – Principal stress; transformation of [stress] tensor components between rotated coordinate frames
  – Special cases: homogeneous stress; plane stress

• Constitutive connections: isotropic linear elasticity
  – Isotropic linear elastic material properties: E, ν, G, and K
  – Stress/strain and strain/stress relations
  – Putting it all together: Navier equations of equilibrium in terms of displacements
  – Boundary conditions and boundary value problems
Geometry of Deformation

- **Origin**: $0$; Cartesian basis vectors, $e_1, e_2, \& e_3$
- **Reference location** of material point: $\mathbf{x}$; specified by its cartesian components, $x_1, x_2, x_3$
- **Displacement vector** of material point: $\mathbf{u}(\mathbf{x})$; specified by displacement components, $u_1, u_2, u_3$
- Each function, $u_i$ ($i=1,2,3$), in general depends on position $\mathbf{x}$ functionally through its components: e.g., $u_1 = u_1(x_1,x_2,x_3)$; etc.
- **Deformed location** of material point: $\mathbf{y}(\mathbf{x})=\mathbf{x}+\mathbf{u}(\mathbf{x})$

\[
\begin{align*}
  \mathbf{x} &= x_1 e_1 + x_2 e_2 + x_3 e_3 \\
  \mathbf{u}(\mathbf{x}) &= u_1 e_1 + u_2 e_2 + u_3 e_3 \\
  \mathbf{y}(\mathbf{x}) &= \mathbf{x} + \mathbf{u}(\mathbf{x}) \\
  &= (x_1+u_1) e_1 + (x_2+u_2) e_2 + (x_3+u_3) e_3
\end{align*}
\]
Displacement of Nearby Points

- Neighboring points: \( x \) and \( x + \Delta x \)
- Displacements: \( u(x) \) and \( u(x + \Delta x) \)
- Deformed: \( y(x) \) and \( y(x + \Delta x) \)
- Displacements: \( u(x) \) and \( u(x + \Delta x) \)
- Vector geometry: \( \Delta y = \Delta x + \Delta u \), where \( \Delta u = u(x + \Delta x) - u(x) \)

\[
\Delta u \equiv u(x + \Delta x) - u(x) = \sum_{i=1}^{3} e_i [u_i(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) - u_i(x_1, x_2, x_3)] \\
\equiv \sum_{i=1}^{3} \Delta u_i e_i
\]
Displacement Gradient Tensor

Taylor series expansions of functions $u_i$:

$$u_i(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) \equiv u_i(x_1, x_2, x_3) + \frac{\partial u_i}{\partial x_1} \Delta x_1 + \frac{\partial u_i}{\partial x_2} \Delta x_2 + \frac{\partial u_i}{\partial x_3} \Delta x_3 + o(\Delta x)$$

$$= u_i(x_1, x_2, x_3) + \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j} \Delta x_j$$

Thus, on returning to the expression on previous the slide, $\Delta u_i$ is given, for each component ($i=1,..3$), by

$$\Delta u_i = \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j} \Delta x_j$$

Components of the displacement gradient tensor can be put in matrix form:

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$
Displacement Gradient and Extensional Strain in Coordinate Directions

Suppose that \( \Delta x = \Delta x_1 e_1 \); then, with \( \Delta y = \Delta x + \Delta u \),

\[
\Delta y = \frac{\Delta x_1 e_1}{\Delta x} + \frac{\partial u_1}{\partial x_1} \frac{\Delta x_1 e_1}{\Delta u} + \frac{\partial u_2}{\partial x_1} \frac{\Delta x_1 e_2}{\Delta u} + \frac{\partial u_3}{\partial x_1} \frac{\Delta x_1 e_3}{\Delta u}.
\]

\[
|\Delta y| = \sqrt{\Delta y \cdot \Delta y} = \sqrt{\Delta x_1} \sqrt{(1 + \frac{\partial u_1}{\partial x_1})^2 + (\frac{\partial u_2}{\partial x_1})^2 + (\frac{\partial u_3}{\partial x_1})^2}.
\]

The fractional change in length (extensional strain) of a material line element initially parallel to \( x_1 \) axis is \( \frac{\partial u_1}{\partial x_1} \); similar conclusions apply for coordinate directions 2 and 3.
Displacement Gradient and Shear Strain

Let \( QR = \Delta x_1 e_1 \) & \( QP = \Delta x_2 e_2 \)

Line segments initially perpendicular
Deformed lines: \( Q'R' \) & \( Q'P' \)

\(|Q'R'| = |\Delta x_1|(1 + \frac{\partial u_1}{\partial x_1})\)
\(|Q'P'| = |\Delta x_2|(1 + \frac{\partial u_2}{\partial x_2})\)

The total reduction in angle of 2 line segments initially perpendicular to coordinate axes 1 and 2 is

\[ \theta_1 + \theta_2 = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \]

\[ \angle P'Q'R' = \pi/2 - (\theta_1 + \theta_2) \]

\[ \sin \theta_1 = \frac{\frac{\partial u_2}{\partial x_1} \Delta x_1}{|Q'R'|} \]
\[ = \frac{\frac{\partial u_2}{\partial x_1}}{(1 + \frac{\partial u_1}{\partial x_1})} \Rightarrow \]

\[ \sin \theta_1 \doteq \theta_1 \doteq \frac{\partial u_2}{\partial x_1} ; \text{ similarly} \]
\[ \sin \theta_2 \doteq \theta_2 \doteq \frac{\partial u_1}{\partial x_2} \]

Similar results apply for all axis pairs
Strain Tensor (I)

The cartesian components of the strain tensor are given, for $i=1..3$ and $j=1..3$, by

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Written out in matrix notation, this index equation is

$$
\begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) \\
\frac{1}{2}(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) \\
\frac{1}{2}(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}) & \frac{\partial u_3}{\partial x_3}
\end{bmatrix}
$$

- Each of the 9 components in the $3 \times 3$ matrices on each side of the matrix equation are equal, so this is equivalent to 9 separate equations.
- The strain tensor is symmetric, in that, for each $i$ and $j$, $\varepsilon_{ij} = \varepsilon_{ji}$.
Strain Tensor (II)

The cartesian components of the strain tensor are given, for $i=1..3$ and $j=1..3$, by

$$
\varepsilon_{ij} \equiv \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
$$

Written out in matrix notation, this index equation is

$$
\begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \\
\frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2} \left( \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) \\
\frac{1}{2} \left( \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \right) & \frac{1}{2} \left( \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right) & \frac{\partial u_3}{\partial x_3}
\end{bmatrix}
$$

- Diagonal components of the strain tensor are the extensional strains along the respective coordinate axes;
- Off-diagonal components of the strain tensor are $\frac{1}{2}$ times the total reduction in angle (from $\pi/2$) of a pair of deformed line elements that were initially parallel to the two axes indicated by the off-diagonal row and column number.
Fractional Volumetric Change

For any values of the strain tensor components, \( \varepsilon_{ij} \), the fractional volume change at a material point, sometimes called the dilatation at the point, is given by

\[
\frac{V_{\text{deformed}} - V_{\text{initial}}}{V_{\text{initial}}} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}
\]

\[
= \sum_{k=1}^{3} \varepsilon_{kk}
\]

This relation holds whether or not the values of \( \varepsilon_{11}, \varepsilon_{22}, \) and \( \varepsilon_{33} \) equal each other, and whether or not any or all of the shear strain components (e.g., \( \varepsilon_{12} = \varepsilon_{21} \)) are zero-valued or non-zero-valued.

The sum of diagonal elements of a matrix of the cartesian components of a tensor is called the trace of the tensor; thus, the fractional volume change is the trace of the strain tensor.
**Strain Deviator Tensor**

Components of the strain deviator tensor, are given in terms of the components of the strain tensor by

\[ \epsilon_{ij}^{(\text{dev})} \equiv \epsilon_{ij} - \frac{1}{3} \delta_{ij} \sum_{k=1}^{3} \epsilon_{kk} \]

Here \( \delta_{ij} \) are components of the Kronecker identity matrix, satisfying \( \delta_{ij} = 1 \) if \( i=j \), and \( \delta_{ij} = 0 \) if \( i \) is not equal to \( j \)

- Off-diagonal components of the strain deviator tensor equal corresponding off-diagonal components of the strain tensor;
- Each diagonal component of the strain deviator tensor differs from the corresponding diagonal component of the strain tensor by \( 1/3 \) of the trace of the strain tensor

**Exercise**: evaluate the trace of the strain deviator tensor.
Strain Decomposition

Alternatively, the strain tensor can be viewed as the sum of

• a shape-changing (but volume-preserving) part (the strain deviator)

Plus

• a volume-changing (but shape-preserving) part (one-third trace of strain tensor times identity matrix):

\[
\varepsilon_{ij} = \varepsilon_{ij}^{\text{(dev)}} + \frac{1}{3} \delta_{ij} \sum_{k=1}^{3} \varepsilon_{kk}
\]

Later, when we look more closely at isotropic linear elasticity, we will find that the two “fundamental” elastic constants are

• the bulk modulus, \( K \), measuring elastic resistance to volume-change, and

• the shear modulus, \( G \), measuring elastic resistance to shape-change
Geometric Aspects of Strain

Undeformed segment:

\( \Delta x \): undeformed vector from P to Q
\( \Delta s \): length of vector = |PQ|
\( e_{(P \rightarrow Q)} \): unit vector pointing in direction from P to Q

\[
\Delta x = \Delta s e_{(P \rightarrow Q)}
\]

\[
\Delta s = |\Delta x| = \sqrt{\Delta x \cdot \Delta x}
\]

\[
e_{(P \rightarrow Q)} = \frac{\Delta x}{\Delta s}
\]

Deformed segment:

\( \Delta y \): deformed vector from P’ to Q’
\( \Delta S \): length of vector = |P’Q’|
\( e_{(P' \rightarrow Q')} \): unit vector pointing in direction from P’ to Q’

\[
\Delta y = \Delta S e_{(P' \rightarrow Q')}
\]

\[
\Delta S = |\Delta y| = \sqrt{\Delta y \cdot \Delta y}
\]

\[
e_{(P' \rightarrow Q')} = \frac{\Delta y}{\Delta S}
\]
Fractional Length Change: Arbitrary Initial Direction

Undeformed:

\[ \Delta x = \Delta s e_{(P\rightarrow Q)} \]

\[ \Delta s = |\Delta x| = \sqrt{\Delta x \cdot \Delta x} \]

\[ m \equiv e_{(P\rightarrow Q)} = \frac{\Delta x}{\Delta s} \]

\[ m = \sum_{i=1}^{3} m_i e_i; \]

\[ m_i = m \cdot e_i; \]

\[ 1 = m \cdot m = m_1^2 + m_2^2 + m_3^2 \]

\[ \Delta x = \Delta s m \iff \Delta x_i = \Delta s m_i \]

Deformed length (squared):

\[ (\Delta S)^2 = |\Delta y|^2 = \Delta y \cdot \Delta y \]

\[ = (\Delta x + \Delta u) \cdot (\Delta x + \Delta u) \]

\[ = \Delta x \cdot \Delta x + \Delta x \cdot \Delta u + \Delta u \cdot \Delta x + \Delta u \cdot \Delta u \]

\[ = (\Delta s)^2 + \Delta s (m \cdot \Delta u + \Delta u \cdot m) + \Delta u \cdot \Delta u \]

\[ = (\Delta s)^2 + (\Delta s) \left( \sum_{i=1}^{3} m_i \Delta u_i + \sum_{j=1}^{3} \Delta u_j m_j \right) + \sum_{i=1}^{3} \Delta u_i \Delta u_i \]

But, \[ \Delta u_i = \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j} \Delta x \]

Finally:

\[ \Delta S = \Delta s \left[ 1 + \sum_{i=1}^{3} \sum_{j=1}^{3} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) m_i m_j \right] \]

\[ \frac{|\Delta y| - |\Delta x|}{|\Delta x|} = \frac{\Delta S - \Delta s}{\Delta s} \]

\[ = \frac{1}{2} \sum_{i=1}^{3} \sum_{i=1}^{3} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) m_i m_j \]
Local Axial Strain in Any Direction

Strain along unit direction \( \mathbf{m} \):

\[
\frac{|\Delta y| - |\Delta x|}{|\Delta x|} = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) m_i m_j
\]

\[
\varepsilon_m = \sum_{i=1}^{3} \sum_{j=1}^{3} \varepsilon_{ij} m_i m_j
\]

Vector components of \( \mathbf{m} \):

\[
\{m_i\} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} (3 \times 1)
\]

\[
[m_i] = \begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix} (1 \times 3)
\]

[Extended] matrix multiplication provides strain in direction parallel to \( \mathbf{m} \):

\[
\varepsilon_m = \begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} (3 \times 1)
\]
Example

Suppose that the components of the strain tensor are

\[
\begin{bmatrix}
\epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\
\epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\
\epsilon_{31} & \epsilon_{32} & \epsilon_{33}
\end{bmatrix} = \begin{bmatrix}
0.003 & -0.001 & 0.002 \\
-0.001 & -0.002 & 0. \\
0.002 & 0. & -0.002
\end{bmatrix}
\]

Find the fractional change in length of a line element initially pointing along the direction \( \mathbf{m} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) / \sqrt{3} \)

**Solution:** equal components \( m_i = 1 / (3)^{1/2} \)

\[
\epsilon_m = \begin{bmatrix}
1 / \sqrt{3} & 1 / \sqrt{3} & 1 / \sqrt{3}
\end{bmatrix} \begin{bmatrix}
0.003 & -0.001 & 0.002 \\
-0.001 & -0.002 & 0. \\
0.002 & 0. & -0.002
\end{bmatrix} \begin{bmatrix}
1 / \sqrt{3} \\
1 / \sqrt{3} \\
1 / \sqrt{3}
\end{bmatrix}
\]

\[
= \frac{1}{3} \times 0.001 = 0.000333
\]
Change of Basis Vectors;  
Change of Components: but  
No Change in Vector

Given:
• a vector \( \mathbf{v} \);
• 2 sets of cartesian basis vectors:
  \{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \} and \{ \mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3' \}
• components of \( \mathbf{v} \) wrt \( \{ \mathbf{e}_i \} \): \( \{ v_i \} \);
• components of \( \mathbf{v} \) wrt \( \{ \mathbf{e}_i' \} \): \( \{ v'_i \} \);

\[
\mathbf{v} = \sum_{i=1}^{3} v_i \mathbf{e}_i = \sum_{j=1}^{3} v'_j \mathbf{e}'_j
\]

**Question:** what relationships exist connecting
The components of \( \mathbf{v} \) in the two bases?
Vector Dot Product and Vector Components

Consider the following dot product operations:

\[ e_1 \cdot v = e_1 \cdot (v_1 e_1 + v_2 e_2 + v_3 e_3) = v_1 \]

\[ e'_2 \cdot v = e'_2 \cdot (v'_1 e'_1 + v'_2 e'_2 + v'_3 e'_3) = v'_2 \]

Evidently, for any basis vector (primed or unprimed)

\[ v_i = e_i \cdot v \]

\[ v'_j = e'_j \cdot v \]

**Thus, any vector** \( \mathbf{v} \) **can be expressed as**

\[ \mathbf{v} = \sum_{i=1}^{3} v_i e_i = \sum_{i=1}^{3} (\mathbf{v} \cdot e_i) e_i \]

\[ \mathbf{v} = \sum_{j=1}^{3} v'_j e'_j = \sum_{j=1}^{3} (\mathbf{v} \cdot e'_j) e'_j \]
Changing Coordinate Systems (I)

Define a matrix $Q_{ij}$ by

$$Q_{ij} \equiv e'_i \cdot e_j$$

$$
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{bmatrix}
= 
\begin{bmatrix}
e'_1 \cdot e_1 & e'_1 \cdot e_2 & e'_1 \cdot e_3 \\
e'_2 \cdot e_1 & e'_2 \cdot e_2 & e'_2 \cdot e_3 \\
e'_3 \cdot e_1 & e'_3 \cdot e_2 & e'_3 \cdot e_3
\end{bmatrix}
$$

Express primed components in terms of unprimed:

$$v'_i = e'_i \cdot v = e'_i \cdot \left( \sum_{j=1}^{3} v_j e_j \right) = \sum_{j=1}^{3} Q_{ij} v_j$$

$$
\begin{bmatrix}
v_1' \\
v_2' \\
v_3'
\end{bmatrix}
= 
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
$$

Alternatively, matrix multiplication to convert vector components:

$$\{v'_i\} = [Q_{ij}] \{v_j\}$$

$$\{v_i\} = [Q_{ij}]^T \{v'_j\}$$

Note: the matrix $[Q_{ij}]$ is said to be **orthogonal**:

- Determinant of $[Q_{ij}] = 1$
- Matrix transpose is matrix inverse:
  $$[Q_{ij}]^{-1} = [Q_{ij}]^T = [Q_{ji}]$$
Changing Coordinate Systems (II)

Define a matrix \( Q_{ij} \) by

\[
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{bmatrix}
\equiv
\begin{bmatrix}
e'_1 \cdot e_1 & e'_1 \cdot e_2 & e'_1 \cdot e_3 \\
e'_2 \cdot e_1 & e'_2 \cdot e_2 & e'_2 \cdot e_3 \\
e'_3 \cdot e_1 & e'_3 \cdot e_2 & e'_3 \cdot e_3
\end{bmatrix}
\]

Express unprimed components in terms of primed:

\[
v_i = e_i \cdot v = e_i \cdot \left( \sum_{j=1}^{3} v'_j e'_j \right) = \sum_{j=1}^{3} Q_{ji} v'_j
\]

\[
\{v'_i\} = [Q_{ij}] \{v_j\}
\]

Matrix multiplication to convert vector components:

\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= 
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{bmatrix}
\begin{bmatrix}
v'_1 \\
v'_2 \\
v'_3
\end{bmatrix}
= 
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{bmatrix}^T
\begin{bmatrix}
v'_1 \\
v'_2 \\
v'_3
\end{bmatrix}
\]

Note: the matrix \([Q_{ij}]\) is said to be **orthogonal**:

- Determinant of \([Q_{ij}] = 1\)
- Matrix transpose is matrix inverse:

\[
[Q_{ij}]^{-1} = [Q_{ij}]^T = [Q_{ji}]
\]
Transformation of Displacement Gradient (Tensor) Components

Vector/vector operation (unprimed components):
\[
\{\Delta u_i\} = \left[ \frac{\partial u_i}{\partial x_j} \right] \{\Delta x_j\}
\]

Pre-multiply by \([Q]\):
\[
\begin{align*}
\{\Delta u'_m\} &= \left[ Q_{mi} \right] \{\Delta u_i\} \\
&= \left[ Q_{mi} \right] \left[ \frac{\partial u_i}{\partial x_j} \right] \{\Delta x_j\} \\
&= \left[ Q_{mi} \right] \left[ Q_{jn} \right]^T \{\Delta x'_n\}
\end{align*}
\]

Substitute on both sides:
\[
\begin{align*}
\{\Delta u'_m\} &= \left[ Q_{mi} \right] \left[ \frac{\partial u_i}{\partial x_j} \right] \left[ Q_{jn} \right]^T \{\Delta x'_n\} \\
&= \left[ \frac{\partial u'_m}{\partial x'_n} \right] \{\Delta x'_n\}
\end{align*}
\]

Vector/vector operation in primed components:
\[
\{\Delta u'_m\} = \left[ \frac{\partial u'_m}{\partial x'_n} \right] \{\Delta x'_n\}
\]

This must always hold so that
\[
\begin{align*}
\left[ \frac{\partial u'_m}{\partial x'_n} \right] &= \left[ Q_{mi} \right] \left[ \frac{\partial u_i}{\partial x_j} \right] \left[ Q_{jn} \right]^T
\end{align*}
\]

This procedure transforms the cartesian components of any second-order tensor, including \(\varepsilon_{ij}\).
Change of Tensor Components with Respect to Change of Basis Vectors

For each primed index, \( i' \) and \( j' \), the tensor component with respect to the primed basis vectors, \( A_{i'j'} \), is given by

\[
A_{i'j'} = \sum_{m=1}^{3} \sum_{n=1}^{3} Q_{i'm} Q_{j'n} A_{mn}
\]

Alternatively, the complete matrix of the primed components of the tensor can be obtained from matrix multiplication:

\[
\begin{bmatrix}
A_{1'1'} & A_{1'2'} & A_{1'3'} \\
A_{2'1'} & A_{2'2'} & A_{2'3'} \\
A_{3'1'} & A_{3'2'} & A_{3'3'}
\end{bmatrix}
= \begin{bmatrix}
Q_{11} & Q_{21} & Q_{31} \\
Q_{12} & Q_{22} & Q_{32} \\
Q_{13} & Q_{23} & Q_{33}
\end{bmatrix}
\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}
= \begin{bmatrix}
Q_{11} & Q_{21} & Q_{31} \\
Q_{12} & Q_{22} & Q_{32} \\
Q_{13} & Q_{23} & Q_{33}
\end{bmatrix}
\]

for any second-order tensor \( \mathbf{A} \)