2.002 MECHANICS & MATERIALS II

INTRODUCTION TO THE MACROSCOPIC THEORY OF PLASTICITY

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RATe-DEPENDENCE AND RATE-INDEPENDENCE

- Input
- Material
- $\sigma - \varepsilon$ curve
- Phenomenon

**Elastic**
- $\dot{\varepsilon}_1, \dot{\varepsilon}_2, \dot{\varepsilon}_3$
- Rate-Independent Response

**Viscous**
- $\dot{\varepsilon}_1, \dot{\varepsilon}_2, \dot{\varepsilon}_3$
- Highly Rate-Dependent Response

**Elastic-Plastic**
- $\dot{\varepsilon}_1, \dot{\varepsilon}_2$
- Slightly Rate-Dependent Response
- Plastic deformation in metals is **thermally-activated** and inherently **rate-dependent**.

- However, the plastic stress-strain response of most single and polycrystalline materials at absolute temperatures $T < (1/3)T_m$, where $T_m$ is the melting temperature of the material in degrees absolute, is only slightly rate-sensitive, and in this temperature regime it may be modeled as **rate-independent**.

<table>
<thead>
<tr>
<th>Material</th>
<th>Melting Temp, °C</th>
<th>$T_m$, °K</th>
<th>$(1/3)T_m$, °K</th>
<th>$\equiv$ °C</th>
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<tbody>
<tr>
<td>Ti</td>
<td>1668</td>
<td>1941</td>
<td>647</td>
<td>374</td>
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<tr>
<td>Fe</td>
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<td>1809</td>
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<tr>
<td>Pb</td>
<td>327</td>
<td>660</td>
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<td>-73</td>
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The governing variables in the one-dimensional rate-independent constitutive model for elastic-plastic solids are

\[ \sigma \quad \text{Stress}, \]
\[ \varepsilon \quad \text{Strain}, \]
\[ \varepsilon^p \quad \text{Plastic strain}, \]
\[ s > 0 \quad \text{Plastic deformation resistance,} \]
\[ \text{internal variable with dimensions of stress,} \]
\[ \text{Initial value of } s: \ s_0 \equiv \sigma_y \quad \text{yield strength in tension} \]
The constitutive model consists of the following set of equations:

1. Elastic strain:

\[ \varepsilon^e = \varepsilon - \varepsilon^p \]

2. Constitutive Equation For \( \sigma \):

\[ \sigma = E [\varepsilon^e] = E [\varepsilon - \varepsilon^p], \quad E \text{ – Young’s Modulus} \]
3. **Yield Condition:**

Let \( s \) denote an internal variable which is a nonzero, positive-valued scalar with the dimensions of stress. We call \( s \) the **deformation resistance**.

The assumption that only a single scalar characterizes the complex internal characteristics of a material is, of course, a gross simplification, but nevertheless, it is widely used to great effectiveness in engineering practice.
Next, we introduce a scalar valued function

\[ f(\sigma, s) = |\sigma| - s, \]

called the **yield function**, and constrain the admissible states \((\sigma, s)\) such that

\[ f(\sigma, s) = |\sigma| - s \leq 0. \]

This is called the **yield condition**.

The set of values of \(\{\sigma\}\) giving resulting in \(f = 0\) for a given \(s\) is called the **yield surface**. In the present one-dimensional context the yield surface is the pair of points \((\sigma = -s, \sigma = +s)\). The yield surface defines the boundary of the elastic domain at a given \(s\).
Note that a state \((\sigma, s)\) with a value \(f(\sigma, s) > 0\) is not admissible. For later use we note that when \(f = 0\), this imposes the restriction that \(\dot{f} \leq 0\).

To see this, let \(f(t) = f(\sigma(t), s(t))\) be the value of \(f\) at time \(t\), and consider a time \(\tau = t + \Delta t\) with \(\Delta t > 0\), then

\[
f(\tau) = f(t) + \dot{f}(t) \Delta t + o(\Delta t).
\]

Since \(f(t) = 0\), we must have \(\dot{f}(t) \leq 0\), otherwise \(f(\tau) > 0\), which is inadmissible.
4. Evolution Equation For $\dot{\epsilon}^p$, Flow Rule:

The plastic strain is taken to evolve according to the flow rule

$$\dot{\epsilon}^p = \dot{\epsilon}^p \text{sign}(\sigma),$$

$$\dot{\epsilon}^p = \begin{cases} 0 & \text{if } f < 0 \quad \text{--- elastic,} \\ 0 & \text{if } f = 0, \quad \text{and } \dot{f} < 0 \quad \text{--- elastic unloading,} \\ > 0 & \text{if } f = 0, \quad \text{and } \dot{f} = 0 \quad \text{--- plastic loading,} \end{cases}$$

$$\text{sign}(\sigma) = \begin{cases} 1 & \text{if } \sigma > 0, \\ -1 & \text{if } \sigma < 0, \end{cases}$$

where $\dot{\epsilon}^p$ is the magnitude of the plastic strain rate, and sign($\sigma$) gives the direction of plastic flow.

The conditions for $\dot{\epsilon}^p \geq 0$ are called the loading/unloading conditions.
\[ \dot{\varepsilon}^p = \begin{cases} 
0 & \text{if } f < 0 \quad \text{— elastic,} \\
0 & \text{if } f = 0, \quad \text{and } \dot{f} < 0 \quad \text{— elastic unloading,} \\
> 0 & \text{if } f = 0, \quad \text{and } \dot{f} = 0 \quad \text{— plastic loading} 
\end{cases} \]

If \( f < 0 \) then \( \dot{\varepsilon}^p = 0 \), the instantaneous response is elastic. If \( f = 0 \), then we have an plastic-state and it is possible that \( \dot{\varepsilon}^p \geq 0 \). If \( f = 0 \) and \( \dot{f} < 0 \), resulting in \( \dot{\varepsilon}^p = 0 \), we have elastic unloading from a plastic state. Finally, if \( f = 0 \) and \( \dot{f} = 0 \), resulting in \( \dot{\varepsilon}^p > 0 \), we have plastic loading.

Since \( \dot{\varepsilon}^p = 0 \) if \( f < 0 \), and \( \dot{\varepsilon}^p > 0 \) is possible only if \( f = 0 \), it follows that \( \dot{\varepsilon}^p f = 0 \):

\[ f \leq 0, \quad \dot{\varepsilon}^p \geq 0, \quad \dot{\varepsilon}^p f = 0. \]
5. **Evolution Equation For \( s \), Hardening Rule:**

Next, the evolution equation for the deformation resistance \( s \) is taken as

\[
\dot{s} = h \dot{\varepsilon}^p, \quad h = \tilde{h}(s)
\]

where \( \tilde{h} \) is a **hardening function**.

The material is said to be **strain-hardening**, **perfectly plastic**, or **strain-softening** according as \( h > 0 \), \( h = 0 \) or \( h < 0 \), respectively.
6. **Consistency Condition:**

Since from a plastic state \( f = 0, \dot{\varepsilon}^p = 0 \) if \( \dot{f} < 0 \), and \( \dot{\varepsilon}^p > 0 \) is possible only if \( \dot{f} = 0 \), we have

\[
\dot{\varepsilon}^p \dot{f} = 0 \quad \text{if} \quad f = 0.
\]

This is called the **consistency (persistency) condition**. To elaborate, in order for \( \dot{\varepsilon}^p > 0 \), a state \((\sigma, s)\) on the boundary of the elastic domain, that is one satisfying \( f(\sigma, s) = 0 \), must persist on the boundary of the elastic domain, so that \( \dot{f}(\sigma, s) = 0 \). That is, during a plastic process the pair \((\sigma, s)\) must continue to satisfy the yield condition \( f = |\sigma| - s = 0 \). This is feasible only if

\[
\dot{f} = |\sigma| - \dot{s} = 0
\]

is satisfied.
7. Magnitude of $\dot{\varepsilon}^p$. Alternate form for the Loading/Unloading Conditions:

The consistency condition serves to determine the magnitude of the plastic strain rate, $\dot{\varepsilon}^p$, when plastic flow occurs.

Since

$$\frac{d |\sigma|}{d\sigma} = \text{sign}(\sigma),$$

we have

$$\dot{|\sigma|} = \text{sign}(\sigma) \dot{\sigma}.$$

Hence, using the rate form of the constitutive equation for stress,

$$\dot{\sigma} = E \left[ \dot{\varepsilon} - \dot{\varepsilon}^p \text{sign}(\sigma) \right]$$
and the evolution equation for $s$,

$$\dot{s} = h \dot{\varepsilon}^p,$$

we have

$$f = |\dot{\sigma}| - \dot{s},$$

$$= \text{sign}(\sigma) \dot{\sigma} - \dot{s},$$

$$= \text{sign}(\sigma) E \left[ \dot{\varepsilon} - \dot{\varepsilon}^p \text{sign}(\sigma) \right] - h \dot{\varepsilon}^p,$$

$$= \text{sign}(\sigma) E \left[ \dot{\varepsilon} - \dot{\varepsilon}^p \right] \{E + h\} \leq 0.$$

We assume that

$$g = \{E + h\} > 0.$$

This is an important assumption; it sets a limit on the negative values of the strain hardening function $h$. 
Thus the magnitude of the plastic strain rate is

\[ \dot{\varepsilon}^p = \begin{cases} 
0 & \text{if } f < 0 \text{ elastic}, \\
0 & \text{if } f = 0, \text{ and } \{\text{sign}(\sigma) E [\dot{\varepsilon}]\} < 0, \\
g^{-1} \{\text{sign}(\sigma) E [\dot{\varepsilon}]\} & \text{if } f = 0, \text{ and } \{\text{sign}(\sigma) E [\dot{\varepsilon}]\} > 0,
\end{cases} \]

where

\[ g \equiv \{E + h\} > 0. \]
Schematic of plastic loading and unloading from a state of stress which satisfies the yield condition

\[ f = |\sigma| - s = 0. \]
8. Elastic-plastic Tangent Moduli:

During plastic loading

\[ \dot{\sigma} = E \left[ \dot{\varepsilon} - \dot{\varepsilon}^p \right] = E \left[ \dot{\varepsilon} - \dot{\varepsilon}^p \text{sign}(\sigma) \right], \]

\[ = E \left[ \dot{\varepsilon} - g^{-1} E \left[ \dot{\varepsilon} \right] \right] = E \left[ \dot{\varepsilon} - \frac{E}{E + h} \dot{\varepsilon} \right] = E \left[ 1 - \frac{E}{E + h} \right] \dot{\varepsilon}. \]

\[ = \left( \frac{E h}{E + h} \right) \dot{\varepsilon}, \]
Hence,

\[ \dot{\sigma} = E^{ep} [\dot{\varepsilon}], \]

with

\[ E^{ep} = \begin{cases} E & \text{if } \dot{\varepsilon}^p = 0, \\ \left( \frac{E h}{E + h} \right) & \text{if } \dot{\varepsilon}^p > 0, \end{cases} \]

is the the \textbf{elastic-plastic tangent modulus}.

This provides an interpretation of our assumption \( g = E + h > 0 \). During plastic loading, for a hardening material \( h > 0 \) and \( E^{ep} > 0 \). For a non-hardening material \( h = 0 \) and \( E^{ep} = 0 \). For a strain-softening material \( h < 0 \) and \( E^{ep} < 0 \), but our assumption \( g = E + h > 0 \) precludes \( E^{ep} = -\infty \).
SUMMARY OF 1-D FORMULATION

Regarding $\epsilon$ as the independent variable and $\{\sigma, \epsilon^p, s\}$ as the dependent variables, the one-dimensional rate-independent constitutive model for elastic-plastic solids with isotropic hardening consists of the following set of equations:

1. Elastic strain:
   \[ \epsilon^e = \epsilon - \epsilon^p. \]

2. Constitutive Equation For $\sigma$:
   \[ \sigma = E [\epsilon - \epsilon^p]. \]
3. Yield Condition:

\[ f = |\sigma| - s \leq 0. \]

4. Flow Rule and Hardening Rule:

\[ \dot{\epsilon}^p = \dot{\epsilon}^p \text{sign}(\sigma), \]
\[ \dot{s} = h \dot{\epsilon}^p, \quad h = \tilde{h}(s). \]

5. Complementarity Conditions and Consistency Condition:

\[ f \leq 0, \quad \dot{\epsilon}^p \geq 0, \quad \dot{\epsilon}^p f = 0. \]
\[ \dot{\epsilon}^p \dot{f} = 0 \quad \text{if} \quad f = 0. \]
6. **Magnitude of the plastic strain rate:**

\[ \dot{\varepsilon}^p = \begin{cases} 
0 & \text{if } f < 0, \\
0 & \text{if } f = 0, \text{ and } \{ \text{sign}(\sigma) E [\dot{\varepsilon}] \} < 0, \\
g^{-1} \{ \text{sign}(\sigma) E [\dot{\varepsilon}] \} & \text{if } f = 0, \text{ and } \{ \text{sign}(\sigma) E [\dot{\varepsilon}] \} > 0, 
\end{cases} \]

with

\[ g \equiv \{ E + h \} > 0. \]
To complete this constitutive model for a given material, the material properties/functions that need to be determined are

1. The Young’s modulus $E$.

2. The initial values $s_0$ of $s$. This is widely called the **yield strength** of the material and denoted by

   $$\sigma_y \equiv s_0.$$ 

3. The strain-hardening function

   $$h = \widehat{h}(s).$$
The hardening function \( \hat{h}(s) \) is determined as follows:

- Assume that \( E \) and the true stress-strain data (\( \sigma \) versus \( \epsilon \)) have been obtained from a compression or a tension test.

- Then using \( \epsilon^p = \epsilon - (\sigma/E) \) the (\( \sigma \) versus \( \epsilon \)) data is converted into (\( \sigma \) versus \( \epsilon^p \)).

If the data is obtained from a compression test, then convert the data into (\( |\sigma| \) versus \( |\epsilon^p| \)).
Next, since $|\sigma| = s$ and $|\varepsilon^p| = \bar{\varepsilon}^p$ during plastic flow, the $(|\sigma| \text{ versus } |\varepsilon^p|)$ data is identical to $(s \text{ versus } \bar{\varepsilon}^p)$ data, from which the desired hardening function can be determined as the slope ($h = \frac{ds}{d\bar{\varepsilon}^p}$ versus $s$).

\[ s \text{ vs } \bar{\varepsilon}^p \text{ for 6061-T6 al alloy at room temp.} \]
3-Dimensional Theory

The governing variables in the three-dimensional theory are

\[
\begin{align*}
\sigma_{ij} & \quad \text{Stress} \\
\varepsilon_{ij} &= (1/2) \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \quad \text{Strain} \\
\varepsilon_{ij}^p & \quad \text{Plastic strain} \\
s & \quad \text{Isotropic deformation resistance, dimensions of stress, } s > 0
\end{align*}
\]
The constitutive model consists of the following set of equations:

**Elastic strain:**

\[ \varepsilon_{ij}^e = \varepsilon_{ij} - \varepsilon_{ij}^p \]

**Constitutive Equation For Stress:**

\[ \sigma_{ij} = \sum_{k,l} C_{ijkl} \left[ \varepsilon_{kl} - \varepsilon_{kl}^p \right]. \]

\[ C_{ijkl} = \frac{E}{2(1+\nu)} \left\{ \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right\} + \frac{E\nu}{(1+\nu)(1-2\nu)} \delta_{ij} \delta_{kl}, \]

- \( E \) — Young’s modulus,
- \( \nu \) — Poisson’s ratio.
Yield Condition:

We introduce a **yield condition** which bounds the levels of stresses in the material. For isotropic materials, a simple yield condition is

\[ f(\sigma, s) \leq 0, \]

where \( f(\sigma, s) \) is a scalar-valued function of the applied stress \( \sigma \), and the scalar \( s \) is a **material property** called the **deformation resistance** of the material.
**Isotropy** requires that the dependence on $\sigma$ in the function $f(\sigma, s)$ can only appear in terms of its principal invariants $\{I_1, I_2, I_3\}$.

Since $\sigma$ is symmetric, then so also is the **stress deviator**

$$\sigma' = \sigma - (1/3)(\text{tr} \sigma)1.$$

The symmetric tensor $\sigma'$ has only five independent components, and only two independent non-zero invariants:

$$J_2 = \frac{1}{2} \left[ \sum_{i,j} \sigma'_{ij} \sigma'_{ij} \right], \quad J_3 = \det \begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix}.$$
Thus, instead of the list of invariants \( \{I_1, I_2, I_3\} \) for the stress, we may use the following as an alternative list of invariants for \( \sigma \):

\[
I_1 = \sum_k \sigma_{kk}, \quad J_2 = \frac{1}{2} \left[ \sum_{i,j} \sigma'_{ij} \sigma'_{ij} \right], \quad J_3 = \text{det} \left[ \sigma' \right].
\]

Additional invariants may be defined in terms of \( \{J_1, J_2, J_3\} \). The following list of invariants for the stress tensor are widely used in the theory of isotropic plasticity:

\[
\bar{p} = -\frac{1}{3} I_1 \quad \bar{\sigma} = \sqrt{\frac{3}{2} \sum_{i,j} \sigma'_{ij} \sigma'_{ij}}
\]

The invariant \( \bar{p} \) is called the mean normal stress, and \( \bar{\sigma} \) is called the equivalent tensile stress.
The invariants $\bar{p}$ and $\bar{\sigma}$ written out in full take the forms:

1. **Mean normal stress:**

   \[ \bar{p} = -\frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}). \]

   In the case of a state of hydrostatic pressure, $\sigma_{ij} = -p \delta_{ij}$, the mean normal stress is $\bar{p} = p$.

2. **Equivalent tensile stress:**

   \[ \bar{\sigma} = \left[ \frac{1}{2} \left\{ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right\} 
     + 3 \left\{ \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \right\} \right]^{1/2} \]

   In the case of pure tension $\sigma_{11} = \sigma$, all other $\sigma_{ij} = 0$, the equivalent tensile stress is $\bar{\sigma} = |\sigma|$. 
Thus we may express our isotropic yield condition as

\[ f(\bar{p}, \bar{\sigma}, s) \leq 0. \]

For ductile metallic polycrystalline materials it has been found experimentally that the function \( f(\bar{p}, \bar{\sigma}, s) \) can, to a very good approximation, be taken to be independent of \( \bar{p} \), and the most widely used yield condition is the Mises yield condition proposed by by Richard von Mises in 1913:

\[ f(\sigma, s) = \bar{\sigma} - s \leq 0, \]

with

\[
\bar{\sigma} = \left[ \frac{1}{2} \left\{ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right\} + 3 \left\{ \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \right\} \right]^{1/2}.
\]
Note that for this yield function

\[ f(\sigma, s) = \bar{\sigma} - s = \sqrt{\frac{3}{2} \sum_{k,l} \sigma'_{kl} \sigma'_{kl}} - s \]

the components of the outward normal to the yield surface, \( f = \bar{\sigma} - s = 0 \), at the current stress point are

\[ \frac{\partial f}{\partial \sigma_{ij}} = \left\{ \frac{3 \sigma'_{ij}}{2 \bar{\sigma}} \right\}. \]
Normality Flow Rule (Levy, Saint Venant):

\[ \dot{\varepsilon}_{ij}^p = \frac{3}{2} \frac{\sigma_{ij}'}{\bar{\sigma}} \]

The quantity

\[ \dot{\varepsilon}^p \equiv \sqrt{\frac{2}{3} \sum_{i,j} \dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p} \geq 0. \]

is called the equivalent tensile plastic strain rate.

Note that since \( \sigma_{ij}' \) is deviatoric, \( \sum_{j=1}^{3} \dot{\varepsilon}_{jj}^p = 0 \). Thus, according to this flow rule, plastic flow is incompressible.
Equivalent tensile plastic strain rate:

$$
\ddot{\varepsilon}^p \equiv \sqrt{\frac{2}{3} \sum_{i,j} \dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p} = \left| \frac{2}{9} \left\{ (\dot{\varepsilon}_{11}^p - \dot{\varepsilon}_{22}^p)^2 + (\dot{\varepsilon}_{22}^p - \dot{\varepsilon}_{33}^p)^2 + (\dot{\varepsilon}_{33}^p - \dot{\varepsilon}_{11}^p)^2 \right\} 
+ \frac{4}{3} \left\{ (\dot{\varepsilon}_{12}^p)^2 + (\dot{\varepsilon}_{23}^p)^2 + (\dot{\varepsilon}_{31}^p)^2 \right\} \right|^{1/2}
$$

Let $\ddot{\varepsilon}_{11}^p$ denote the plastic strain rate in a simple tension/compression test. Then because of isotropy $\ddot{\varepsilon}_{22}^p = \ddot{\varepsilon}_{33}^p$, and because of plastic incompressibility

$$
\ddot{\varepsilon}_{11}^p + \ddot{\varepsilon}_{22}^p + \ddot{\varepsilon}_{33}^p = 0 \implies \ddot{\varepsilon}_{22}^p = \ddot{\varepsilon}_{33}^p = -\left( \frac{1}{2} \right) \ddot{\varepsilon}_{11}^p,
$$

and all other $\ddot{\varepsilon}_{k\ell}^p = 0$. Under these conditions

$$
\ddot{\varepsilon}^p = \left| \ddot{\varepsilon}_{11}^p \right|,
$$

and hence the name equivalent tensile plastic strain rate.
The quantity
\[ \bar{\varepsilon}^p(t) = \int_0^t \dot{\varepsilon}^p(\xi) \, d\xi, \]
is called the equivalent tensile plastic strain.

**Hardening Rule:**
\[ \dot{s} = h \dot{\varepsilon}^p, \quad h = \tilde{h}(s) \text{ hardening function} \]

**Complementarity Conditions and Consistency Condition:**
\[ f \leq 0, \quad \bar{\varepsilon}^p \geq 0, \quad \bar{\varepsilon}^p f = 0, \]
\[ \dot{\varepsilon}^p \dot{f} = 0 \quad \text{if} \quad f = 0. \]
Magnitude of the plastic strain rate:

\[
\varepsilon_p = \begin{cases} 
0 & \text{if } f < 0, \\
0 & \text{if } f = 0 \text{ and } \left\{ \sum_{p,q} n_{pq} \dot{\sigma}_{pq}^{\text{trial}} \right\} < 0, \\
0 & \text{if } f = 0 \text{ and } \left\{ \sum_{p,q} n_{pq} \dot{\sigma}_{pq}^{\text{trial}} \right\} = 0, \\
\sqrt{\frac{3}{2}} g^{-1} \left\{ \sum_{p,q} n_{pq} \dot{\sigma}_{pq}^{\text{trial}} \right\}, & \text{if } f = 0 \text{ and } \left\{ \sum_{p,q} n_{pq} \dot{\sigma}_{pq}^{\text{trial}} \right\} > 0, 
\end{cases}
\]

with

\[
g \equiv \left[ \frac{3E}{2(1+\nu)} + h \right] > 0,
\]

\[n_{pq} \equiv \sqrt{\frac{3}{2}} \left( \frac{\sigma'_{pq}}{\sigma} \right)\] outward unit normal to yield surface,

\[\dot{\sigma}_{pq}^{\text{trial}} \equiv \sum_{r,s} C_{pqrs} \dot{\varepsilon}_{rs}\] trial stress rate.
Summary

Strain Rate in Terms of Stress Rate:

\[
\dot{\varepsilon}_{ij} = \frac{1}{E} \left[ (1 + \nu)\dot{\sigma}_{ij} - \nu \left( \sum_k \dot{\sigma}_{kk} \right) \delta_{ij} \right] + \sqrt{\frac{3}{2}} \dot{\varepsilon}^p n_{ij}
\]

\[
\dot{\varepsilon}^p = \begin{cases} 
0 & \text{if } f < 0, \\
0 & \text{if } f = 0 \text{ and } \left\{ \sum_{p,q} n_{pq} \dot{\sigma}_{pq}^{\text{trial}} \right\} < 0, \\
0 & \text{if } f = 0 \text{ and } \left\{ \sum_{p,q} n_{pq} \dot{\sigma}_{pq}^{\text{trial}} \right\} = 0, \\
\sqrt{\frac{3}{2}} g^{-1} \left\{ \sum_{p,q} n_{pq} \dot{\sigma}_{pq}^{\text{trial}} \right\}, & \text{if } f = 0 \text{ and } \left\{ \sum_{p,q} n_{pq} \dot{\sigma}_{pq}^{\text{trial}} \right\} > 0,
\end{cases}
\]
with Mises yield condition

\[ f(\sigma, s) = \bar{\sigma} - s \leq 0 \]
\[ \bar{\sigma} = \left[ \frac{1}{2} \left\{ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right\} \right]^{1/2} + 3 \left\{ \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \right\}^{1/2}, \]

and

\[ g \equiv \left[ \frac{3E}{2(1 + \nu)} + h \right] > 0, \]
\[ n_{pq} \equiv \sqrt{\frac{3}{2}} \left( \frac{\sigma'_{pq}}{\bar{\sigma}} \right) \] outward unit normal to yield surface,
\[ \dot{\sigma}_{pq}^{\text{trial}} \equiv \sum_{r,s} C_{pqrs} \dot{\epsilon}_{rs} \] trial stress rate.
Stress Rate in Terms of Strain Rate:

\[ \dot{\sigma}_{ij} = \sum_{k,l} \mathcal{L}_{ijkl} \dot{\epsilon}_{kl} \]

\[ \mathcal{L}_{ijkl} = \begin{cases} \frac{C_{ijkl}}{C_{ijkl} - (3/2) g^{-1} (m_{ij} m_{kl})} & \text{if } \dot{\epsilon}^p = 0, \\ \frac{E}{2(1+\nu)} \left\{ \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right\} + \frac{E \nu}{(1+\nu)(1-2\nu)} \delta_{ij} \delta_{kl}, & \text{if } \dot{\epsilon}^p > 0, \end{cases} \]

\[ C_{ijkl} = \frac{E}{2(1+\nu)} \left\{ \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right\} + \frac{E \nu}{(1+\nu)(1-2\nu)} \delta_{ij} \delta_{kl}, \]

\[ g = \left[ \frac{3E}{2(1+\nu)} + h \right] > 0, \]

\[ m_{ij} = \sum_{k,l} C_{ijkl} n_{kl}, \quad n_{ij} = \sqrt{\frac{3}{2}} \left( \frac{\sigma'_{ij}}{\bar{\sigma}} \right) \]

\[ \mathcal{L}_{ijkl} \text{ are the elasto-plastic tangent moduli, and } C_{ijkl} \text{ are the elastic moduli.} \]
Hardening Rule:

\[ \dot{s} = h \dot{\varepsilon}^p, \quad h = \hat{h}(s) \text{ hardening function} \]

This is a generalization of the one-dimensional case with \( \dot{\varepsilon}^p \) the equivalent tensile plastic strain rate.

The quantity

\[ \bar{\varepsilon}^p(t) = \int_0^t \dot{\varepsilon}^p(\xi) \, d\xi, \]

is the equivalent tensile plastic strain.

For monotonic proportional loading the evolution equation for \( s \) may be integrated to give \( s \) as a function of \( \bar{\varepsilon}^p \):

\[ s = \hat{s}(\bar{\varepsilon}^p). \]
To complete this constitutive model for a given material, the material properties/functions that need to be determined are the elastic moduli

$$(E, \nu),$$

the initial value

$$s_0 \equiv \sigma_y \quad \text{yield strength}$$

of the deformation resistance $s$, and the hardening function

$$h = \hat{h}(s).$$
The hardening function \( \bar{h}(s) \) is determined from a **simple tension/compression** test as follows:

- Assume that \( E \) and the true stress-strain data (\( \sigma \) versus \( \epsilon \)) have been obtained from a compression or a tension test.

- Then using \( \epsilon^p = \epsilon - (\sigma/E) \) the (\( \sigma \) versus \( \epsilon \)) data is converted into (\( \sigma \) versus \( \epsilon^p \)).

If the data is obtained from a compression test, then convert the data into (\( |\sigma| \) versus \( |\epsilon^p| \)).
Next, since $|\sigma| = s$ and $|\epsilon^p| = \overline{\epsilon^p}$ during plastic flow, the ($|\sigma|$ versus $|\epsilon^p|$) data is identical to ($s$ versus $\overline{\epsilon^p}$) data, from which the desired hardening function can be determined as the slope ($h = \frac{ds}{d\overline{\epsilon^p}}$ versus $s$).

$s$ vs $\overline{\epsilon^p}$ for 6061-T6 al alloy at room temp.
Concluding Remarks

- The equations for elastic-plastic deformation are coupled differential evolutions for the stress $\sigma_{ij}$ and the deformation resistance $s$.

- The solution of complex boundary-value problems using these equations is best carried out numerically.

- The constitutive equations described here (with some possible change in notation and terminology) are the ones most widely used in modern commercial finite-element programs.