Example 2 (continued)

Figure 1: A spring attached to a cart with an attached pendulum. Figure by MIT OCW.

Figure 2: Free body diagram of spring, cart, and pendulum system. Figure by MIT OCW.
Coordinate System: $x$, $\theta$: Generalized coordinates. Chosen to describe system well.

**Kinematics**

$r_A = x \hat{i}$  \quad $r_B = (x + L \sin \theta)\hat{i} - L \cos \theta \hat{j}$
$r_A = \ddot{x} \hat{i}$  \quad $r_B = (\dot{x} + L \dot{\theta} \cos \theta)\hat{i} + L \dot{\theta} \sin \theta \hat{j}$
$\dot{r}_A = \ddot{x} \hat{i}$  \quad $\dot{r}_B = (\ddot{x} + L \ddot{\theta} \cos \theta - L \dot{\theta}^2 \sin \theta)\hat{i} + (L \dot{\theta} \sin \theta + L \dot{\theta}^2 \cos \theta)\hat{j}$

Do not want to introduce unknown forces.

**Kinetics**

Linear Momentum in x direction

\[-k x = M \ddot{x} + m \ddot{x} + m L \dot{\theta} \cos \theta - m L \dot{\theta}^2 \sin \theta\]  \hspace{1cm} (1)

$(F_{spring} = F_{M,x} + F_{m,x})$

Need another equation: Angular momentum for this case. Could also use conservation of energy.

Angular Momentum: Choose $A$ because only $mg$ has moment about $A$.

$\tau_A = \frac{d}{dt} \mathbf{H}_A + \mathbf{v}_A \times \mathbf{P}$

$\tau_A = -mgL \sin \theta \hat{k}$  \hspace{1cm} (2)

No moment for $M$ about $A$ because $A$ is the center of mass of $M$.

\[
\mathbf{H}_A = \mathbf{AB} \times m \dot{\mathbf{B}}
\]

\[
= (L \sin \theta \dot{i} - L \cos \theta \dot{j}) \times m [(\ddot{x} + L \dot{\theta} \cos \theta)\hat{i} + (L \dot{\theta} \sin \theta)\hat{j}]
\]

\[
= (m L^2 \ddot{\theta} + m L \ddot{x} \cos \theta) \hat{k}
\]  \hspace{1cm} (3)

$$\mathbf{v}_A \times \mathbf{P} = \dot{x} \hat{j} \times (M \ddot{x} + m \ddot{x} + L \dot{\theta} \cos \theta)\hat{i} + m(L \dot{\theta} \sin \theta)\hat{j}
\]

\[
= mL \dot{\theta} \sin \theta \hat{k}
\]  \hspace{1cm} (4)

Notice: All torques in $\hat{k}$ direction.

$$\mathbf{\omega} = \frac{d}{dt} \mathbf{\Theta} + \mathbf{\Omega}$$

Substitute and simplify.

\[m L^2 \ddot{\theta} + m L \ddot{x} \cos \theta + mgL \sin \theta = 0\]  \hspace{1cm} (5)
Discussion
Now we have 2 equations in 2 unknowns. How do we solve? Simulate with MATLAB. This system has certain vibrations.

Equations are nonlinear.

Examples of Linear Terms: $\dot{x}$, $\dot{\theta}$, $\ddot{x}$, $\theta$, $x$, $\theta$

Combinations of variables: Nonlinear

Operations of variables: $\cos \theta$, $\sin \theta$, $\theta^2$, $\dot{\theta}^2$ (Nonlinear)

In Equation 1 Nonlinear terms are $L\dot{\theta} \cos \theta$ and $-L\dot{\theta}^2 \sin \theta$
In Equation 5 Nonlinear terms are $mL\ddot{x} \cos \theta$ and $mgL \sin \theta$

Equation 1 and Equation 5 contain intricate dynamics.

1965: Edward Lorenz at MIT - made a breakthrough in equations predicting weather. Ran simulations on 3 equations.

He could never get the same results twice. Uncertainty with initial conditions, especially due to vacuum tubes used then.

Any small uncertainties can be amplified by equations. “Butterfly effect.”


Simulation
To simulate, reorganize equations 1 and 5. First rewrite 5 as

$$\ddot{\theta} = -\frac{1}{L}(\ddot{x} \cos \theta + g \sin \theta)$$

Then substitute into Equation 1

$$\ddot{x}(M + m + m \cos^2 \theta) + mg \sin \theta \cos \theta - mL\ddot{\theta} \sin \theta + kx = 0 \quad (6)$$

Use Equation 5 to substitute for $\ddot{x}$ in Equation 1 and obtain:

$$\ddot{\theta}(mL^2) + mL \cos \theta \left(\frac{mL\ddot{\theta} \sin \theta - kx - mg \sin \theta \cos \theta}{M + m + m \cos^2 \theta}\right) + mgL \sin \theta = 0 \quad (7)$$

To solve these numerically:

$$x_1 = x, y_1 = \theta, x_2 = \dot{x} = \dot{x}_1, y_2 = \dot{\theta} = \dot{y}_1$$
2 Second Order Equations → 4 First Order Equations

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{(M + m + m \cos^2 y_1)} \left[-mg \sin y_1 \cos y_1 + mLy^2_2 \sin y_1 - kx_1 \right] \\
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -\frac{\cos y_1}{L} \left[ \frac{mLy^2_2 \sin y_1 - kx_1 - mg \sin y_1 \cos y_1}{M + m + m \cos^2 y_1} \right] - \frac{g}{L} \sin y_1
\end{align*}
\]

General Form:

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}
\]

where \( f_1, f_2, f_3, \) and \( f_4 \) are functions of \( x_1, x_2, y_1, \) and \( y_2. \) Set initial conditions for \( x_1, x_2, y_1, \) and \( y_2. \) Matlab can solve right-hand side for next time.

Simplest is Euler step-method for solving.

In MATLAB, you will use:

\texttt{ode45}

Rest of course: Will have some mathematical analysis of the equations of motion to acquire understanding separate from MATLAB.