Velocity and Acceleration of a Point in a Moving Frame

The figure below shows a point $B$ in a coordinate system $Axyz$ which is translating and rotating with respect to a fixed coordinate system $OXYZ$.

**Position**

The position of point $B$ with respect to the fixed frame is given by

$$O_{LB} = O_{LA} + A_{LB}$$

(1)

**Velocity**

The velocity of point $B$ with respect to the fixed frame is given by

$$O\mathbf{V}_B = O\mathbf{V}_A + A\mathbf{V}_B|^{\omega_A=0} + O\mathbf{\omega}_A \times A_{LB}$$

(2)

where

- $O\mathbf{V}_A$ is the velocity of the origin of the moving coordinate system
- $A\mathbf{V}_B|^{\omega_A=0}$ is the velocity of point $B$ with respect to point $A$ as seen by an observer attached to the moving coordinate system, i.e. were the moving coordinate system not rotating
- $O\mathbf{\omega}_A \times A_{LB}$ is the velocity of point $B$ with respect to point $A$ due to the rotation of the moving coordinate system
Acceleration

The acceleration of \( B \) with respect to the fixed frame is given by

\[
^{O}a_B = \overset{O}{a}_A + \overset{A}{a}_B|_{\omega_A=0} + 2\overset{O}{\omega}_A \times \overset{A}{v}_B + \overset{O}{\omega}_A \times \overset{A}{r}_B + \overset{O}{\omega}_A \times (\overset{O}{\omega}_A \times \overset{A}{r}_B)
\]  

(3)

where

- \( \overset{O}{a}_A \) is the acceleration of the origin of the moving coordinate system
- \( \overset{A}{a}_B|_{\omega_A=0} \) is the acceleration of point \( B \) as seen by an observer attached to the rotating coordinate system
- \( 2\overset{O}{\omega}_A \times \overset{A}{v}_B \) is the Coriolis acceleration
- \( \overset{O}{\omega}_A \times \overset{A}{r}_B \) is the Eulerian acceleration
- \( \overset{O}{\omega}_A \times (\overset{O}{\omega}_A \times \overset{A}{r}_B) \) is the centripetal acceleration
Velocity and Acceleration of a Point in a Plane - Polar Coordinates

The figure below shows a point $P$ moving along a path in a fixed coordinate system $OXYZ$. The path is described in polar coordinates, $r$ and $\theta$, with unit vectors $\hat{r}$, $\hat{\theta}$ in the $r$-direction and $\theta$-direction, respectively.

![Diagram of point moving along a path in a plane with polar coordinates](image)

**Position**

The point’s position, $^{O}_{L}P$, is given by

$$^{O}_{L}P = r\hat{r}$$

(1)

Note that $^{O}_{L}P$ is a vector with magnitude $r$ in the $\hat{r}$ direction.

**Velocity**

The velocity of point $P$ with respect to the fixed frame is obtained by differentiation.

$$^{O}_{L}V_{P} = \dot{v} = ^{O}_{L} \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\ddot{r}$$

It can be shown that the time derivatives of the unit vectors, $\dot{\hat{r}}$ and $\ddot{\hat{\theta}}$, rotating with angular velocity $\dot{\theta}$ are

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} \quad \text{and} \quad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$$

So the velocity of point $P$ is given by

$$\dot{v} = (\dot{r})\hat{r} + (r\dot{\theta})\hat{\theta}$$

(2)
or

\[ v = v_r \hat{r} + v_\theta \hat{\theta} \quad \text{where} \quad v_r = \dot{r} \quad \text{and} \quad v_\theta = r \dot{\theta} \]

\[ \hat{v} = \hat{v}_r + \hat{v}_\theta \]

\[ \dot{v} = (\ddot{r} - r \dot{\theta}^2) \hat{r} + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \hat{\theta} \]  

\( O \hat{a}_P = \hat{a} \)

\[ \hat{a} = a_r \hat{r} + a_\theta \hat{\theta} \]

where

\[ a_r = \ddot{r} - r \dot{\theta}^2 \quad \text{and} \quad a_\theta = r \ddot{\theta} + 2 \dot{r} \dot{\theta} \]
Amusement Park Ride - Problem Statement

A ride at an amusement park consists of four symmetrically located seats, driven to rotate about the vertical axis $A$ at a constant rate of $\omega_2$ rev/min, with respect to the supporting arm, $r_1$. The vertical axis, $O$, is driven by another motor at a constant rate of $\omega_1$ rev/min, with respect to ground. All four seats are a distance, $r_2$, from the $A$ axis and the $O$ and $A$ axes are parallel as shown in the figure. Assume the existence of a fixed coordinate system, $OXYZ$, attached to the ground and a rotating coordinate system, $Axyz$, attached to the component holding the passengers as shown the figure. Find the acceleration with respect to ground of each of the four passengers at the instant shown.

Amusement Park Ride - Solution - Accelerations

The general expression for the acceleration of $B$ with respect to the fixed frame is given by

$$ O \ddot{a}_B = O \ddot{a}_A + \dot{A} \ddot{a}_B|_{\omega_A=0} + 2(\dot{O} \omega_A \times \dot{A} \ddot{r}_B) + \dot{O} \omega_A \times A \ddot{r}_B + \omega_A \times (\dot{O} \omega_A \times A \ddot{r}_B) \quad (1) $$

Correspondence

- point $B$ in equation (1) will correspond to each of the passengers at points 1 through 4, in turn
- the angular velocity of the moving frame, $\dot{O} \omega_A = \omega_1 + \omega_2$
The first term, $O_{a_A}$, describes the acceleration of point A with respect to frame O as shown below.

![Acceleration Diagram](attachment:image.png)

or

$$O_{a_A} = [-r_1 \omega_1^2 \cos \theta] \hat{i} + [-r_1 \omega_1^2 \sin \theta] \hat{j}$$

The second term, $A_{a_{B}}|_{\omega_A=0}$, is zero because the point of interest is not accelerating within the moving frame, i.e. $A_{a_{B}} = 0$.

The third term, $2(O_{\omega_A} \times A_{v_{B}})$, is zero because the point of interest is not translating within the moving frame, i.e. $A_{v_{B}} = 0$.

The fourth term, $O_{\omega_A} \times A_{\omega_{B}}$, is zero because the angular accelerations are zero.

Only the last term, $O_{\omega_A} \times (O_{\omega_A} \times A_{\omega_{B}})$, i.e. the centripetal acceleration, is non-zero as shown below.

![Centripetal Acceleration Diagram](attachment:image.png)

or

$$A_{a_1} = -r_2(\omega_1 + \omega_2)^2 \hat{i}$$

$$A_{a_2} = -r_2(\omega_1 + \omega_2)^2 \hat{j}$$

$$A_{a_3} = r_2(\omega_1 + \omega_2)^2 \hat{i}$$

$$A_{a_4} = r_2(\omega_1 + \omega_2)^2 \hat{j}$$

so

$$O_{a_1} = [-r_1 \omega_1^2 \cos \theta - r_2(\omega_1 + \omega_2)^2] \hat{i} + [-r_1 \omega_1^2 \sin \theta] \hat{j}$$

$$O_{a_2} = [-r_1 \omega_1^2 \cos \theta] \hat{i} + [-r_1 \omega_1^2 \sin \theta - r_2(\omega_1 + \omega_2)^2] \hat{j}$$

$$O_{a_3} = [-r_1 \omega_1^2 \cos \theta + r_2(\omega_1 + \omega_2)^2] \hat{i} + [-r_1 \omega_1^2 \sin \theta] \hat{j}$$

$$O_{a_4} = [-r_1 \omega_1^2 \cos \theta] \hat{i} + [-r_1 \omega_1^2 \sin \theta + r_2(\omega_1 + \omega_2)^2] \hat{j}$$
Block Sliding on Rotating Bar - Problem Statement

The bar $OA$ rotates about a horizontal axis through $O$ with a constant counterclockwise velocity $\omega = 3$ rad/sec, in the presence of gravity. As it passes the position $\theta = 0$, a small block of mass, $m$, is placed on it at a radial distance $r = 18$ in. If the block is observed to slip at $\theta = 50^\circ$,

1. determine the coefficient of static friction, $\mu_s$, between the block and the bar.

2. determine the torque applied to the bar at $O$.

Block Sliding on Rotating Bar - Solution

The free body diagram, depicting the external forces acting on the mass is shown in the figure below.
Note:

- the unit vectors \( \hat{r} \) and \( \hat{\theta} \)
- the mass’s weight (i.e. the force gravity exerts on the mass), \( F_w = mg \)
- the normal force (of the bar on the mass), \( F_n \)
- the friction force (of the bar on the mass), \( F_f = \mu_s F_n \)

**Accelerations**

In polar coordinates, the acceleration of the mass is given by

\[
a = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}
\]

(1)

or

\[
a = (-r\omega^2)\hat{r} + (0)\hat{\theta}
\]

(2)

**Newton’s Laws**

Applying Newton’s Laws in the \( \hat{r} \) and \( \hat{\theta} \) directions,

\[
\Sigma F_r = ma_r
\]

\[
\Sigma F_\theta = ma_\theta
\]

we know that

\[
\mu_s F_n - mg\sin\theta = -mr\omega^2
\]

\[
F_n - mg\cos\theta = 0
\]

**Coefficient of Static Friction**

Eliminating \( F_n \), we find that

\[
\mu_s = \tan\theta - \frac{r\omega^2}{g\cos\theta}
\]

\[
\mu_s = 0.54
\]
Torque

The free-body diagram for the bar is

Summing torques in the $\hat{k}$ direction,

$$\tau - mg\cos\theta = 0$$

The torque applied to the bar at $O$ is

$$\tau = rF_n\hat{k} = r(mg\cos\theta)\hat{k}$$