Reading:

- Class Handout: *Modeling Part 2: Summary of One-Port Primitive Elements*
- Nise: Secs. 2.4 and 2.6.

1 Rotational Systems (continued)

Example 1

The diagram shows a mechanical tachometer that uses frictional drag plates to create a torque proportional to angular velocity difference. The angular velocity is indicated by the displacement $\theta$ of a torsional spring.

\[
\begin{array}{c}
\text{frictional drag plates} \\
\hline
\end{array}
\]

Find the transfer function relating the displacement of the indicator $\theta$ to the input angular velocity $\Omega_{\text{in}}$

\[
H(s) = \frac{\theta(s)}{\Omega_{\text{in}}(s)}
\]

and show that for a constant input angular velocity the steady-state indicated speed $\theta_{\text{ss}} \propto \Omega_{\text{i}}$.

Solution: There are two distinct angular velocities, and the system graph is:
Using impedances, redraw the graph combining the inertia and the spring into a single impedance $Z_2$:

![Graph combining inertia and spring into a single impedance](chart.png)

Then

$$\Omega_J = \frac{Z_2}{Z_1 + Z_2} \Omega_{in}(s)$$

$$= \frac{s/(Js^2 + K)}{1/B + s/(Js^2 + K)} \Omega_{in}(s)$$

$$= \frac{B}{Js^2 + Bs + K} \Omega_{in}(s).$$

But the angular displacement $\theta(s) = \Omega_J(s)/s$ so that

$$H(s) = \frac{B}{Js^2 + Bs + K}$$

If the input velocity is a step function at $t = 0$, the $\Omega_{in}(s) = \Omega/s$ and the steady-state indicated value will be

$$\theta_{ss} = \lim_{t \to \infty} \theta(t) = \lim_{s \to 0} s\theta(s) = \lim_{s \to 0} sH(s) \frac{\Omega}{s} = \frac{B}{K} \Omega$$

show that the indicated speed ($\theta$) is proportional to the input angular velocity.

(This example is repeated using mesh equations below.)
2 Transfer Function Generation Using Mesh/Loop Equations

This method is useful when the system contains an across variable source.

We will develop the method using the following mechanical system as an example.

The input is a velocity source \( V(t) \), and the output is the velocity of the mass \( m_2 \).

**Step 1:** (This step is optional.) Do some simple impedance combinations to reduce the complexity, being careful not to eliminate any nodes or branches that specify the output variable.

**Step 2:** Define a set of closed loops/meshes, making sure that each branch is covered by at least one loop.

**Step 3:** Write a loop equation for each loop:

\[
\begin{align*}
v Z_1 + v Z_2 - V &= 0 \\
v Z_3 + v Z_4 - v Z_2 &= 0 \\
v Z_5 - v Z_3 - v Z_1 &= 0
\end{align*}
\]
Step 4: At this point we assume that each loop has a continuous through variable with it, that is loop $n$ has $F_n$ associated with it. When a branch is contacted by more than one loop, the branch through variable is the sum of all of the contacting loop through variables. We use impedance relationships and substitute loop through variable expressions for the across variables:

$$Z_1(F_1 - F_3) + Z_2(F_1 - F_2) - V = 0$$
$$Z_3(F_2 - F_3) + Z_4F_2 - Z_2(F_1 - F_2) = 0$$
$$Z_5F_3 - Z_3(F_2 - F_3) - Z_1(F_1 - F_3) = 0$$

Step 5: Rewrite these equations collecting terms in the loop through variables to create a set of linear equations:

$$(Z_1 + Z_2)F_1 - Z_2F_2 - Z_1F_3 = V$$
$$-Z_2F_1 + (Z_2 + Z_3 + Z_4)F_2 - Z_3F_3 = 0$$
$$-Z_1F_1 + Z_3F_2 + (Z_1 + Z_3 + Z_5)F_3 = 0$$

It may be useful to express these equations in matrix form:

$$\begin{bmatrix}
(Z_1 + Z_2) & -Z_2 & -Z_1 \\
-Z_2 & (Z_2 + Z_3 + Z_4) & -Z_3 \\
-Z_1 & Z_3 & (Z_1 + Z_3 + Z_5)
\end{bmatrix} \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix} = \begin{bmatrix}
V \\
0 \\
0
\end{bmatrix}$$

Step 6: Identify how the output variable is related to the loop through variables. In this case

$$v_{out} = F_2Z_4$$

Step 7: Solve the set of linear equations for the variable(s) identified in Step 6, in this case $F_2$, using any method, and substitute in the output equation (Step 6).

(we will not do this step here).

Example 2

Repeat the mechanical tachometer problem of Example 1, this time using mesh equations.
With the system graph and two loops defined as above, the loop equations are:

\[
\begin{aligned}
\Omega_B + \Omega_J - \Omega_{in} &= 0 \\
\Omega_K - \Omega_J &= 0
\end{aligned}
\]

Substitute for the across variables (\(\Omega\)):

\[
\begin{aligned}
Z_B T_1 + Z_K (T_1 - T_2) &= \Omega_{in} \\
Z_J T_2 - Z_K (T_1 - T_2) &= 0
\end{aligned}
\]

Rearrange:

\[
\begin{aligned}
(Z_B + Z_K) T_1 - Z_K T_2 &= \Omega_{in} \\
-Z_K (T_1 + (Z_J + Z_K) T_2) &= 0
\end{aligned}
\]

and in matrix form

\[
\begin{bmatrix}
Z_B + Z_K & -Z_K \\
-Z_K & Z_J + Z_K
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2
\end{bmatrix}
= \begin{bmatrix}
\Omega_{in} \\
0
\end{bmatrix}
\]

The output variable is \(\Omega_J = Z_J T_2\), therefore use Cramer’s rule to solve for \(T_2\).

\[
T_2 = \frac{\begin{vmatrix}
Z_B + Z_K & \Omega_{in} \\
-Z_K & 0
\end{vmatrix}}{\begin{vmatrix}
Z_B + Z_K & -Z_K \\
-Z_K & Z_J + Z_K
\end{vmatrix}} = \frac{Z_K \Omega_{in}}{Z_B Z_J + Z_B Z_K + Z_K Z_J}
\]

and since \(\Omega_J = Z_J T_2\), \(Z_J = 1/(Js)\), \(Z_K = s/K\), and \(Z_B = 1/B\)

\[
\Omega_J = \frac{Z_J Z_K \Omega_{in}}{Z_B Z_J + Z_B Z_K + Z_K Z_J} = \frac{Bs}{Js^2 + Bs + K} \Omega_{in}
\]

and as before, \(\theta(s) = \Omega_J/s\) giving

\[
H(s) = \frac{\theta(s)}{\Omega_{in}(s)} = \frac{B}{Js^2 + Bs + K}
\]

3 Transfer Function Generation Using Node Equations

This method is useful when the system contains a through variable source.

We will demonstrate the method using the following graph:
**Step 1:** Write node equations for each node (except the reference node) - let $f$ be a generalized through variable.

- Node (a):
  \[
  F_s - f_{Z_1} - f_{Z_2} = 0 \quad \text{node(a)}
  \]
  \[
  f_{Z_2} - f_{Z_4} - f_{Z_3} = 0 \quad \text{node(b)}
  \]

**Step 2:** Express the branch through variables in terms of admittances ($Y = 1/Z$) and across variables

- Across variables:
  \[
  (v_a - v_b)Y_2 + (v_a - v_c)Y_1 = F_s
  \]
  \[
  (v_a - v_b)Y_2 + (v_b - v_c)Y_4 - (v_b - v_c)Y_3 = 0
  \]

**Step 3:** Collect terms in the across variables, and form a set of linear equations (note that node (c) is the reference node, therefore $v_c = 0$).

- Linear equations:
  \[
  (Y_1 + Y_2)v_a - Y_2v_b = F_s
  \]
  \[
  Y_2v_a - (Y_2 + Y_3 + Y_4)v_b = 0
  \]

**Step 4:** Identify the output variable in terms of nodal across variables and solve the linear equations.

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**Example 3**

Find $H(s) = X_{m_2}(s)/F(s)$ for the following system:

The graph has been simplified by combining parallel elements:
At the nodes (a) and (b):

\[-F_B - F_{m_1} - F_s = 0\]
\[F_B - F_{m_2} + F_s = 0\]

Substitute for through variables

\[-(v_a - v_b)Y_3 - v_a Y_1 = F_s\]
\[(v_a - v_b)Y_3 - v_b Y_1 = -F_s\]

Reorganize and write in matrix form:

\[
\begin{bmatrix}
-(Y_3 + Y_1) & Y_3 \\
Y_3 & -(Y_2 + Y_3)
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b
\end{bmatrix}
= \begin{bmatrix}
F_s \\
-F(s)
\end{bmatrix}
\]

Recognize that the output \(x_{m_2}(s) = \frac{v_b(s)}{s}\), use Cramer’s rule to solve for \(v_b\):

\[
v_b = \frac{\begin{vmatrix}
-(Y_3 + Y_1) & F(s) \\
Y_3 & -F_s
\end{vmatrix}}{Y_1 Y_2 + Y_3(Y_1 + Y_2)} = \frac{Y_1 F_s}{Y_1 Y_2 + Y_3(Y_1 + Y_2)}
\]

\(Y_1 = m_1s, Y_2 = m_2s,\) and \(Y_3 = B + K/s\), giving

\[
v_{m_2}(s) = v_b(s) = \frac{m_1 s}{m_1 m_2 s^2 + B(m_1 + m_2)s + K(m_1 + m_2)} F_s(s)
\]

and

\[
H(s) = \frac{x_{m_2}(s)}{F_s(s)} = \frac{1}{s} \frac{v_{m_2}(s)}{F_s(s)} = \frac{m_1}{m_1 m_2 s^2 + B(m_1 + m_2)s + K(m_1 + m_2)}
\]