Problem 1. Pendulum mounted on elastic support. A collar of mass $m$ slides without friction on a horizontal rigid rod and is restrained by a pair of identical springs with spring constant $k$. A pendulum consisting of a uniform rigid bar of length $L$ and mass $M$ is suspended from the collar by a frictionless pivot.

Figure 1: Pendulum supported on spring-restrained mount.

(a) Select a complete and independent set of generalized coordinates for this system.

(b) Derive differential equations of motion for these coordinates.
Problem 2. Stabilization of rocker. A rocker is machined into the shape shown from a rectangular block of metal of size $2R \times 3R \times h$, where $h$ is the uniform height normal to the sketch. The uniform density of the material is $\rho$, mass per unit volume.

![Figure 2: Dimensions of rocker.](image)

(a) Express the mass $M$ and the centroidal moment of inertia $I_C$ of the rocker in terms of the parameters $\rho$, $R$, and $h$. Some helpful information is summarized in Fig.3. In (i) the centroidal moment of inertia of a uniform disk or cylinder is $I_C = \frac{1}{2} m_1 r^2$. In (ii) the centroidal moment of inertia of a uniform rectangular plate is $I_C = \frac{1}{12} m_2 (a^2 + b^2)$. In (iii) the centroid of a semi-circle is located a distance $\bar{y} = \frac{4}{3\pi} R$ above the base.

![Figure 3: Useful facts about circular and rectangular shapes.](image)

(b) If the rocker is constrained to roll without slipping on the floor, the upright position shown in Fig.2 is an equilibrium position. Is this a stable equilibrium?

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(c) To stabilize the rocker, it is proposed to apply a horizontal force $f(t)$ to the centroid of the rocker, as shown in Fig.4. Derive a differential equation which describes how the rocking angle $\theta$ responds to the excitation $f(t)$.

\[ C \theta f(t) g \]

Figure 4: Force $f(t)$ is applied to rocker.

(d) Linearize the result of (c) by making the approximations $\sin \theta \approx \theta$ and $\cos \theta \approx 1.0$. Transform the differential equation in the time domain into a transfer function from $F(s)$ to $\Theta(s)$ in the Laplace $s$-domain.

(e) It is proposed to construct the force $f(t)$ by observing the angle $\theta$, comparing it with a desired angle $\theta_d$ and using the difference to generate (by means of a linkage driven by a motor) the force

\[ f(t) = K(\theta_d - \theta) \]

where $K$ is the effective gain, with the dimensions of force per radian. Transform this relation to the $s$-domain and couple it to the result of (d). Obtain the poles of the closed-loop transfer function from $\Theta_d(s)$ to $\Theta(s)$. For what range of values of the gain $K$ is the closed-loop system stable?
Problem 3. Eigenvalue problem. The two masses slide without friction on the horizontal rigid rod, and are held in place by two springs with spring constant $k$.

\[ \begin{align*} &\text{Figure 5: Mass-spring vibratory system} \\ &\text{(a) Formulate equations of motion for } x_1(t) \text{ and } x_2(t) \text{ in the form of a matrix differential equation.} \\ &\text{(b) Derive an eigenvalue problem of the form} \\ &\quad [K]\{a\} = \omega^2[M]\{a\} \\ &\quad \text{for natural modes of the form} \\ &\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1\sin(\omega t + \phi) \\ a_2\sin(\omega t + \phi) \end{bmatrix} \\ &\text{(c) Solve analytically for the mode shapes } \{a_1 \ a_2\}^T \text{ and the eigenvalues } \omega_1^2 \text{ and } \omega_2^2. \text{ Construct the modal matrix } [\Phi] \text{ whose columns are the mode shape vectors.} \\ &\text{(d) Open MATLAB and type the command: } \text{help eig} \text{ to learn about MATLAB’s eigenvalue capabilities. Apply the command } [V, \ D] = \text{EIG} (K, M) \text{ and compare MATLAB’s solution to your solution in (c) above.} \end{align*} \]