In problems 1–3, you will explore the characteristics of the root locus. The root locus is the trajectory of the closed-loop pole as a gain $K$ increases. The closed-loop transfer function is $KG(s)/(1 + KG(s)H(s))$, and the closed-loop poles are the roots of $1 + KG(s)H(s) = 0$ (i.e., the roots of the denominator of the closed-loop transfer function). In this problem set, we deal with unity-feedback only, which implies $H(s) = 1$. Thus every pole on the root locus should satisfy $1 + KG(s) = 0$. Because $s$ is a complex number, $KG(s) = -1$ leads to two requirements:

$$K = 1/|G(s)|,$$

$$\angle KG(s) = (2n + 1) \times 180^\circ \quad (n \text{ is an arbitrary integer}).$$

You can use these relations either geometrically or algebraically.

1. For the complex number $s_1 = -1 + j$,

   a. The phase of the complex number $(s_1 + 2)(s_1 + 0)$.

   Answer:

   From the above figure,

   $$\angle (s_1 + 2)(s_1 + 0) = \angle (s_1 + 2) + \angle (s_1 + 0)$$

   $$= \frac{3\pi}{4} + \frac{3\pi}{4} = \pi.$$

   Algebraically,

   $$(s_1 + 2)(s_1 + 0) = (-1 + j + 2)(-1 + j) = (1 + j)(-1 + j) = -2,$$
and
\[ \angle(-2) = \pi. \]

b. The value of the real number \( K \) such that \( K|s_1 + 2||s_1 + 0| = 1 \).

Answer: From the above figure,
\[ K \cdot \sqrt{2} \cdot \sqrt{2} = 1 \Rightarrow K = 1/2. \]

Algebraically,
\[ |s_1 + 2| = |1 + j| = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad |s_1| = |-1 + j| = \sqrt{1^2 + 1^2} = \sqrt{2}. \]

Hence,
\[ K \cdot \sqrt{2} \cdot \sqrt{2} = 1 \Rightarrow K = 1/2. \]

c. Does \( s_1 \) belong to the root locus?

Answer: From the problem statement, the open-loop transfer function is given by
\[ G(s) = \frac{1}{s(s + 2)}. \]

From result (a) we determine that \( \angle G(s_1) = \pi \). Therefore, \( s_1 \) is on the root locus. To find the value of gain that would drive the closed-loop pole to location \( s_1 \) on the complex plane, we must satisfy
\[ K \frac{1}{(s_1 + 0)(s_1 + 2)} = 1 \Rightarrow K \frac{1}{\sqrt{2}\sqrt{2}} = 1 \Rightarrow K = 2. \]

Note that result (b) is not directly applicable!

d. MATLAB result

Answer:

\[ 2. \text{ The open loop transfer function } G(s) = 1/\{s(s + 1)(s + 2)\}. \]
a. Show that ± j√2 belongs to the root locus.

Answer:

For \( s = \pm j\sqrt{2} \) to be on the root locus, it must satisfy \( \angle \{s(s + 1)(s + 2)\} \big|_{s=\pm j\sqrt{2}} = \pi \).

\[
\angle \{s(s + 1)(s + 2)\} \bigg|_{s=\pm j\sqrt{2}} = \\
\angle s \bigg|_{s=\pm j\sqrt{2}} + \angle (s + 1) \bigg|_{s=\pm j\sqrt{2}} + \angle (s + 2) \bigg|_{s=\pm j\sqrt{2}} = \frac{\pi}{2} + \theta_1 + \theta_2 = \pi,
\]

\[\Rightarrow \theta_1 + \theta_2 = \frac{\pi}{2},\]

where \( \theta_1 = \tan^{-1}(\sqrt{2}) \) and \( \theta_2 = \tan^{-1}(\sqrt{2}/2) \). Hence, if \( \theta_1 + \theta_2 = \pi/2 \) so that cot(\( \theta_1 + \theta_2 \)) = 0, then \( s = \pm j\sqrt{2} \) is on the root locus.

\[
\cot(\theta_1 + \theta_2) = \frac{1}{\tan(\theta_1 + \theta_2)} = \frac{\cos(\theta_1 + \theta_2)}{\sin(\theta_1 + \theta_2)} = \frac{\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2}{\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2} = 0,
\]

which requires that \( \cos \theta_1 \cos \theta_2 = \sin \theta_1 \sin \theta_2 \). From the geometric relation,

\[
\cos \theta_1 \cos \theta_2 = \left( \frac{1}{\sqrt{3}} \right) \left( \frac{2}{\sqrt{6}} \right) = \left( \frac{\sqrt{2}}{\sqrt{3}} \right) \left( \frac{\sqrt{2}}{\sqrt{6}} \right) = \sin \theta_1 \sin \theta_2,
\]

which is true in this case. Therefore \( s = \pm j\sqrt{2} \) belongs to the root locus.

(Or you can simply compute these angles with a calculator and verify that \( \theta_1 + \theta_2 = \pi/2 \).)

b. Compute the feedback gain \( K \).

Answer: On the root locus, \( K = 1/|G(s)| \). Hence,

\[
K = \frac{1}{|G(s)|} \bigg|_{s=\pm j\sqrt{2}} = \sqrt{2}\sqrt{3}\sqrt{6} = 6.
\]
c. Verify algebraically.
   Answer: If \( s = \pm j\sqrt{2} \) belongs to the root locus, then it should satisfy \( 1 + KG(s) = 0 \).

   \[
   1 + K \frac{1}{s(s+1)(s+2)} \bigg|_{s=\pm j\sqrt{2}} = 1 + K \frac{1}{(\pm j\sqrt{2})(1 \pm j\sqrt{2})(2 \pm j\sqrt{2})} = \\
   1 + K \frac{1}{\pm j\sqrt{2}(\pm j3\sqrt{2})} = 1 + K \frac{1}{-6} = 0.
   \]

   Thus \( K = 6 \).

d. What will happen if \( K \) exceeds the value that you computed in question (b)?
   Answer: If \( K > 6 \), then the poles cross over to the right–hand half–plane. The system becomes unstable.

e. Sketch the root locus.
   Answer:
   - Since we have three poles \( \Rightarrow \) the RL has 3 branches
   - The RL has two real–axis segments: one between \( s = 0 \) and \( s = -1 \), the other one between \( s = -2 \) and negative infinity. You would expect to have a breakaway point between \( s = 0 \) and \( s = -1 \) since these are both real poles and a RL real–axis segment lies between them.
   - The asymptotes: The system has three finite poles and no finite zero. Thus you would expect three zeros at infinity, which means that the RL must have three asymptotes.

\[
\sigma_a = \frac{-2-1}{3-0} = -1, \\
\theta_a = \frac{(2m+1)\pi}{3-0} = \left\{ \frac{\pi}{3}, \pi, \frac{5\pi}{3} \right\}.
\]

- Locating the break–in/away points:

\[
K = -\sigma(\sigma + 1)(\sigma + 2),
\]

\[
\frac{dK}{d\sigma} = -(\sigma + 1)(\sigma + 2) - \sigma(\sigma + 2) - \sigma(\sigma + 1) = \\
- [\sigma^2 + 3\sigma + 2 + \sigma^2 + 2\sigma + \sigma^2 + \sigma] = -(3\sigma^2 + 6\sigma + 2) = 0
\]

\[
\sigma = \frac{-3 \pm \sqrt{9 - 2 \times 3}}{3} = \frac{-3 \pm \sqrt{3}}{3}.
\]
Since there is no real-axis segment between $s = -2$ and $s = -1$, $s = \frac{-3 - \sqrt{3}}{3}$ is not a break-in/away point. $s = \frac{-3 + \sqrt{3}}{3}$ is break-away point because it lies between two poles ($s = 0$ and $s = -1$). We had expected a break-away point somewhere in this segment (see bullet #2 above.)

• check the exact point somewhere in this segment (see bullet #2 above.)

f. Matlab root locus.

3. The open loop transfer function $G(s) = \frac{1}{(s + 1)(s + 2)}$.

a. Sketch the root locus

Answer:

• 2 poles ⇒ 2 branches
• One real-axis segment exists in between $s = -1$ and $s = -2$. We expect a break-away point between these two poles
• Asymptotes: The system has two finite poles and no finite zero. Thus you would expect two zeros at infinity, which means the RL has two asymptotes.

$$\sigma_a = \frac{-1 - 2}{2 - 0} = \frac{-3}{2},$$

$$\theta_a = \frac{(2m + 1)\pi}{2 - 0} = \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}.$$ 

• Break-in/away points

$$K = -(\sigma + 1)(\sigma + 2) = 0,$$
$$\frac{dK}{d\sigma} = -\sigma + 2 - (\sigma + 1) = -2\sigma - 3 = 0,$$

$$\Rightarrow \sigma = \frac{3}{2}.$$ 

• The breakaway point and asymptotes’ real-axis intercept $\sigma_a$ coincide with each other. Therefore, the RL lies exactly on top of the asymptote.
b. The closed–loop poles that yield 16.3% OS.

*Answer:*

The damping ratio $\zeta$ that yields 16.3% OS is computed by

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = 0.5.$$  

$$\cos \theta = 0.5 \Rightarrow \theta = 60^\circ.$$  

You draw the line whose angle is 60° to the negative real–axis; the intersection between this line and the root locus is the closed–loop pole that yields 16.3% OS. From the root locus, the real part of the pole is $-2$. Hence, geometrically $p_0 = -3/2 + j3/2 \times \tan(60°) = -3/2 + j3\sqrt{3}/2$. Since the system is second order, this answer is exact.

c. The settling time.

*Answer:*

From the previous result in (b), the absolute value of the real–part of the pole is $\sigma_d = \zeta \omega_n = 3/2$ and the imaginary part is $\omega_d = \omega_n \sqrt{1 - \zeta^2} = \sqrt{7}$. Hence, the settling time is approximately $t_s = \frac{\ln(0.01)}{\sigma_d} = \frac{\ln(0.01)}{3/2} = \frac{-2}{3}$. Therefore, the settling time is approximately $t_s = 5.2$ seconds.
3√3/2. Since the settling time $T_s \approx 4/(\zeta\omega_n)$, you find

$$T_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} = \frac{4}{3/2} = \frac{8}{3}.$$ 

Note that the computed settling time agrees with the MATLAB result.

d. Using geometrical arguments and calculations, compute the value of the gain $K$.

Answer: From the figure drawn in question (c), $|p_0 + 2| = \sqrt{7}$ and $|p_0 + 1| = \sqrt{7}$. Thus we find

$$K = |p_0 + 2||p_0 + 1| = \sqrt{7} \times \sqrt{7} = 7.$$

e. Using a PD controller, achieve the settling time to 75% of the value obtained in question (c) while maintaining the same overshoot.

Answer: The new settling time is 2(= 8/3 × 75%) (s). Thus the absolute value of the real part $\sigma'_d$ of the new pole is 2. To maintain the same %OS, the pole should have the same damping ratio, which means the angle $\theta$ remains the same. So the imaginary part $\omega'_d$ of the pole should be $2 \times \tan(60^\circ) = 2\sqrt{3}$.

The new pole is at $s_1 = -2 + j2\sqrt{3}$. To achieve a RL that overlaps the $s_1$ location, we cascade to the open–loop TF a zero at $s = z_0$ (cascading a zero to the open–loop TF constitutes the PD controller.) Now we must determine the location $z_0$ of the zero.
Denoting $\angle (s_1 + z_0) = \psi$, $|s_1 + z_0| = l_1$, $\angle (s_1 + 2) = \pi/2$, $|s_1 + 2| = l_2$, $\angle (s_1 + 1) = 180 - \alpha$, and $|s_1 + 1| = l_3$, we want to make $s_1$ belong to the root locus. First we apply $\angle KG(s) = 180^\circ$.

$$90^\circ + (180^\circ - \alpha) - \psi = 180^\circ \quad \Rightarrow \quad \psi = 90^\circ - \alpha.$$ 

From the geometry,

$$\tan \alpha = \frac{2\sqrt{3}}{1} = 2\sqrt{3}.$$ 

Thus $\alpha = 73.8979^\circ$. Substituting it in $\psi = 90^\circ - \alpha$, we find

$$\psi = 90^\circ - 73.8979^\circ = 16.1021^\circ.$$ 

From the geometry,

$$\tan \psi = \frac{2\sqrt{3}}{z_0 - 2} \Rightarrow z_0 = 2 + \frac{2\sqrt{3}}{\tan \psi} = 2 + \frac{2\sqrt{3}}{0.2887} = 14.$$ 

Thus, the requisite PD compensator is $(s + 14)$. To find the gain $K$,

$$\frac{Kl_1}{l_2l_3} = 1 \Rightarrow K = \frac{l_2l_3}{l_1} = \frac{2\sqrt{3}\sqrt{1 + (2\sqrt{3})^2}}{\sqrt{12^2 + (2\sqrt{3})^2}} = \frac{2\sqrt{3}\sqrt{13}}{\sqrt{156}} = 1.$$ 

Note that MATLAB’s `sisotool` uses different notation for the compensator, you have to re-calculate a gain $K_z$ to use MATLAB’s `sisotool` as follows:

$$K(s + z_0) = K_z\left(\frac{s}{z_0} + 1\right) \Rightarrow K_z = 14.$$ 

f. Sketch the root locus of the PD-compensated system.

*Answer:*
• 2 poles ⇒ 2 branches
• Two real–axis segments: one in between \( s = -1 \) and \( s = -2 \), and the other in between \( s = -14 \) and negative infinity. We expect a break–away point between the two poles at \(-1, -2\).
• Asymptotes: The system has two finite poles and one zero. You would expect one zero at infinity, which means that the RL has one asymptote.

\[
\theta_a = \frac{(2m + 1)\pi}{2 - 1} = \{\pi\}.
\]

So the asymptote is the part of the real axis towards \(-\infty\).
• The break–in/away points

\[
K = -\frac{(\sigma + 1)(\sigma + 2)}{\sigma + 14},
\]

\[
\frac{dK}{d\sigma} = -\frac{(2\sigma + 3)(\sigma + 14) - (\sigma^2 + 3\sigma + 2)}{(\sigma + 14)^2} = \frac{(2\sigma^2 + 31\sigma + 42) - (\sigma^2 + 3\sigma + 2)}{(\sigma + 14)^2} = \frac{\sigma^2 - 28\sigma + 40}{(\sigma + 14)^2} = 0,
\]

\[
\sigma^2 - 28\sigma + 40 = 0
\]

\[
\sigma = 14 \pm \sqrt{14^2 - 40} = 14 \pm \sqrt{156} = \{26.49, 1.51\}
\]

The first solution is a break–in point (because it lies on the real axis segment of the RL between the zero at \(-14\) and the zero at infinity) and the second solution is a break–away point (because it lies on the real axis segment of the RL between the poles at \(-1, -2\)).

**g.** Verify numerically using MATLAB.

*Answer:*

Answer:

• \( sI - A \)

\[
sI - A = s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} s-3 & -1 \\ 1 & s-3 \end{pmatrix}
\]

• \( AB \)

\[
AB = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 9 & 2 \\ 7 & 16 \end{pmatrix}
\]

Note that \( BA \neq AB \).

Matrices do not commute.

• \( B^{-1}A \)

\[
B^{-1} = \frac{1}{2 \cdot 5 - 3 \cdot (-1)} \begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix},
\]

\[
B^{-1}A = \frac{1}{13} \begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 14 & 8 \\ -11 & 3 \end{pmatrix}.
\]

• \( Bp \)

\[
Bp = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.
\]

• \( Aq \)

\[
Aq = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} e^{-3t} \cos t \\ e^{-3t} \sin t \end{pmatrix} = \begin{pmatrix} 3e^{-3t} \cos t + e^{-3t} \sin t \\ -e^{-3t} \cos t + 3e^{-3t} \sin t \end{pmatrix}.
\]
5. The compensated 2.004 Tower system.

a) Forces acting on the tower.

Answer:

- Inertia force: \(-m_1\ddot{x}_1(t)\)
- Spring force: \(-k_1x_1(t) + k_2(x_2(t) - x_1(t))\)
- Damping force: \(-b_1\dot{x}_1(t) + b_2(\dot{x}_2(t) - \dot{x}_1(t))\)
- Wind force: \(w(t)\)
- Actuator force: \(-a(t)\)

Applying force balance, we obtain an equation of motion for the tower.
\[m_1\ddot{x}_1(t) + (b_1 + b_2)\dot{x}_1(t) - b_2\ddot{x}_2(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = w(t) - a(t).\]

b) Forces acting on the slider.

Answer:

- Inertia force: \(-m_2\ddot{x}_2(t)\)
- Spring force: \(-k_2(x_2(t) - x_1(t))\)
- Damping force: \(-b_2(\dot{x}_2(t) - \dot{x}_1(t))\)
- Actuation force: \(a(t)\)

Applying force balance, we obtain an equation of motion for the slider.
\[m_2\ddot{x}_2(t) + b_2(\dot{x}_2(t) - \dot{x}_1(t)) + k_2(x_2(t) - x_1(t)) = a(t).\]

c) The equations of motion in terms of the state variables.

Answer: Setting four state variables \(\{x_1, v_1, x_2, v_2\}\) and omitting time dependency \((t)\) for simplicity, we can re-write the equations of motion as follows:

\[m_1\dot{v}_1 + (k_1 + k_2)x_1 + (b_1 + b_2)v_1 - k_2x_2 - b_2v_2 = -a + w,\]
\[m_2\dot{v}_2 - k_2x_1 - b_2v_1 + k_2x_2 + b_2v_2 = a.\]

Substituting \(\{x_1, v_1, x_2, v_2\}\) to \(\{q_1, q_1, q_2, q_2\}\), we obtain the equations of motion for the state variables as follows:

\[m_1\dot{q}_2 + (k_1 + k_2)q_1 + (b_1 + b_2)q_2 - k_2q_3 - b_2q_4 = w - a,\]
\[m_2\dot{q}_4 - k_2q_1 - b_2q_2 + k_2q_3 + b_2q_4 = a.\]

d) Solve the equations of motion for \(\dot{q}_2\) and \(\dot{q}_4\).

Answer:

\[
\dot{q}_2 = -\frac{(k_1 + k_2)}{m_1}q_1 - \frac{(b_1 + b_2)}{m_1}q_2 + \frac{k_2}{m_1}q_3 + \frac{b_2}{m_1}q_4 - \frac{1}{m_1}a + \frac{1}{m_1}w, \\
\dot{q}_4 = \frac{k_2}{m_2}q_1 + \frac{b_2}{m_2}q_2 - \frac{k_2}{m_2}q_3 - \frac{b_2}{m_2}q_4 + \frac{1}{m_2}a
\]
e) State-space representation

*Answer:*

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-(k_1 + k_2)/m_1 & -(b_1 + b_2)/m_1 & k_2/m_1 & b_2/m_1 \\
0 & 0 & 0 & 1 \\
k_2/m_2 & b_2/m_2 & -k_2/m_2 & -b_2/m_2
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
0 \\
-1/m_1 \\
0 \\
1/m_2
\end{pmatrix},
\]

and

\[
G = \begin{pmatrix}
0 \\
1/m_1 \\
0 \\
0
\end{pmatrix}.
\]