Summary: Root Locus sketching rules

Negative Feedback

• **Rule 1:** # branches = # poles
• **Rule 2:** symmetrical about the real axis
• **Rule 3:** real-axis segments are to the left of an *odd* number of real-axis finite poles/zeros
• **Rule 4:** RL begins at poles, ends at zeros
• **Rule 5:** Asymptotes: real-axis intercept $\sigma_a$, angles $\theta_a$

\[
\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\# \text{finite poles} - \# \text{finite zeros}} \quad \theta_a = \frac{(2m + 1)\pi}{\# \text{finite poles} - \# \text{finite zeros}} \quad m = 0, \pm 1, \pm 2, \ldots
\]

• **Rule 6:** Real-axis break-in and breakaway points

Found by setting $K(\sigma) = -\frac{1}{G(\sigma)H(\sigma)}$ (\(\sigma\) real) and solving $\frac{dK(\sigma)}{d\sigma} = 0$ for real $\sigma$.

• **Rule 7:** Imaginary axis crossings (*transition to instability*)

Found by setting $KG(j\omega)H(j\omega) = -1$ and solving

\[
\begin{cases} 
\Re\left[KG(j\omega)H(j\omega)\right] = -1, \\
\Im\left[KG(j\omega)H(j\omega)\right] = 0.
\end{cases}
\]

• **Today’s Goal:** Shaping the transient response by adjusting the feedback gain
Damping ratio and pole location

Recall 2nd-order underdamped system

\[ \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \]

Complex poles \(-\sigma_d \pm j\omega_d\),

where \( \begin{cases} \sigma_d &= \zeta\omega_n, \\ \omega_d &= \sqrt{1 - \zeta^2}\omega_n. \end{cases} \)

From the geometry,

\[ \tan \theta = \frac{\sqrt{1 - \zeta^2}}{\zeta} \Rightarrow \]

\[ \cos \theta = \zeta. \]

The angle \(\theta\) that a complex pole subtends to the origin of the s-plane determines the damping ratio \(\zeta\) of an underdamped 2nd order system.

The distance from the pole to the origin equals the natural frequency.
Transient response and pole location

Images removed due to copyright restrictions.


- **Settling time**
  
  \[ T_s \approx \frac{4}{(\zeta \omega_n)}; \]

- **Damped osc. frequency**
  
  \[ \omega_d = \sqrt{1 - \zeta^2 \omega_n} \]

- **Overshoot %OS**
  
  \[ \%OS = \exp \left( - \frac{\zeta \pi}{\sqrt{1 - \zeta^2}} \right) \]

  \[ \tan \theta = \frac{\sqrt{1 - \zeta^2}}{\zeta} \]
Trends in underdamped response as $\zeta$ increases

As $\zeta \uparrow$,

- Rise time $T_r \uparrow$ (slower);
- Settling time $T_s \approx 4/(\zeta\omega_n)$ $\uparrow$ (slower);
- Peak time $T_p = \pi/(\sqrt{1 - \zeta^2}\omega_n)$ $\uparrow$ (slower);
- Overshoot $\%OS \downarrow$ (smaller)

Figure by MIT OpenCourseWare.

Images removed due to copyright restrictions.

Achieving a desired transient with a given RL

As $\zeta \uparrow \iff \theta \downarrow$,

- Rise time $T_r \uparrow$ (slower);
- Settling time $T_s \uparrow$ (slower);
- Peak time $T_p \uparrow$ (slower);
- Overshoot $\%OS \downarrow$ (smaller)

If the given RL does not allow the desired transient characteristics to be achieved, then we must modify the RL by adding poles/zeros (compensator design).
Example: 2\textsuperscript{nd} order – type 1 system

We are given $\zeta = 1/\sqrt{2} = 0.7071$. For this value,

$$\%OS = \exp\left(\frac{\zeta \pi}{\sqrt{1 - \zeta^2}}\right) \times 100 = e^{-\pi} \times 100 = 4.32\%.$$ 

Also, $\cos \theta = \zeta \Rightarrow \theta = \pm 45^\circ$.

We can locate the closed-loop poles by finding the intersection of the root locus with the lines $\theta = \pm 45^\circ$.

We can also estimate the feedback gain $K$ that will yield the required closed-loop poles $-p_{c+}$, $-p_{c-}$ from the relationship

$$K = \frac{|p_{c\pm}|^2}{\frac{0.3162}{|p_{c\pm} + 2|}} = \frac{\sqrt{2} \times \sqrt{2}}{0.3162} = 6.325.$$ 

The numerator is computed geometrically from the equilateral triangle $\{(-2), (p_{c+}), (0)\}$.
Example: higher order system

\[
\frac{K}{s(s+3)(s+7)(s+8)} + \frac{(s+30)}{(s^2+20s+200)}
\]
Positive feedback: sketching the Root Locus

\[\text{Closed-loop TF}(s) = \frac{KG(s)}{1 - KG(s)H(s)}.\]

- **Rule 1:** # branches = # poles
- **Rule 2:** symmetrical about the real axis
- **Rule 3:** real-axis segments are to the left of an *even* number of real-axis finite poles/zeros
- **Rule 4:** RL begins at poles, ends at zeros
- **Rule 5:** Asymptotes: real-axis intercept \(\sigma_a\), angles \(\theta_a\)
  \[\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\# \text{finite poles} - \# \text{finite zeros}}\]
  \[\theta_a = \frac{2m\pi}{\# \text{finite poles} - \# \text{finite zeros}}\]
  \[m = 0, \pm 1, \pm 2, \ldots\]
- **Rule 6:** Real-axis break-in and breakaway points
  Found by setting \(K(\sigma) = + \frac{1}{G(\sigma)H(\sigma)}\) (\(\sigma\) real) and solving \(\frac{dK(\sigma)}{d\sigma} = 0\) for real \(\sigma\).
- **Rule 7:** Imaginary axis crossings (*transition to instability*)
  Found by setting \(KG(j\omega)H(j\omega) = +1\) and solving
  \[
  \begin{cases}
  \text{Re} [KG(j\omega)H(j\omega)] &= +1, \\
  \text{Im} [KG(j\omega)H(j\omega)] &= 0.
  \end{cases}
  \]
Example: positive feedback

\[ R(s) + \frac{K(s + 3)}{s(s + 1)(s + 2)(s + 4)} C(s) \]

\[ K < 0 \iff \]

with \( K > 0 \).

Real-axis asymptote intercept:

\[ \sigma_a = \frac{(-1 - 2 - 4) - (-3)}{4 - 1} = \frac{-4}{3} \]

Asymptote angles

\[ \theta_a = \frac{2m\pi}{4 - 1}, \quad m = 0, 1, 2, \ldots \]

\[ = 0, \quad m = 0, \]

\[ = \frac{2\pi}{3}, \quad m = 1, \]

\[ = \frac{4\pi}{3}, \quad m = 2. \]

Breakaway point:

found numerically.

Image removed due to copyright restrictions.

Please see Fig. 8.26b in Nise, Norman S. Control Systems Engineering. 4th ed. Hoboken, NJ: John Wiley, 2004.