2.019 Design of Ocean Systems

Lecture 9

Ocean Wave Environment

March 7, 2011
Ocean Surface Wave Generation

Waves important to offshore structure design and operation: Wind waves or gravity waves with wave period $T = 5 \sim 20$ seconds, wavelength $O(10)m$ to $O(500)m$.

- Source of forcing: wind
- Source of restoring: gravity
- Source of damping: wave breaking and viscous effects

- When wind starts (0.5 ~ 2.0 knots), capillary waves form (e.g. $V_p = 24$ cm/s → $\lambda = 1.73$ cm)
- As wind becomes stronger, waves become longer

Wind energy input into water:

Frictional drag
Separation drag
Bernoulli effect
• Nonlinear wave-wave interactions cause energy to be transferred into longer waves
• Certain distance and duration (for wind to blow) are necessary for effective energy transfer
• Equilibrium sea: when energy input from the wind is balanced by dissipation
• When wind input energy is larger than dissipation, waves grow
• When wind input energy is smaller than dissipation, waves decay. Short waves decay faster.

Amplitude decays as $e^{-\gamma t}$

$\gamma = 2\nu k^2 = 2\nu \frac{\omega^4}{g^2}$

(Landau + Lifshitz)

Shorter waves are steeper, and easier to break
• Wind must blow over long periods of time and large distances to reach fully-developed state.
• At fully-developed state, $U_w \sim V_p$ (i.e. $\omega_{\text{limit}} \sim g/U_w$)
• Swell: waves are not generated by local wind
• Sea: waves generated by local wind

Required fetch and storm duration:

<table>
<thead>
<tr>
<th>Beaufort scale</th>
<th>wind speed (mph)</th>
<th>fetch (miles)</th>
<th>duration (h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-4</td>
<td>12</td>
<td>15</td>
<td>3</td>
</tr>
<tr>
<td>5-6</td>
<td>25</td>
<td>100</td>
<td>12</td>
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<td>35</td>
<td>400</td>
<td>28</td>
</tr>
<tr>
<td>9</td>
<td>50</td>
<td>1,050</td>
<td>50</td>
</tr>
</tbody>
</table>
Standard Wave Spectra

Based on measured spectra and theoretical results, standard spectrum forms have been developed:

Bretschneider spectrum:

\[ S(\omega) = \frac{1.25}{4} \frac{\omega_m^4}{\omega^5} H_s^2 \exp\{-1.25 \left(\frac{\omega_m}{\omega}\right)^4\} \]

- \(\omega_m\) is peak or modal frequency
- \(H_s\) is significant wave height

\[ \int_0^\infty S(\omega)d\omega = M_0 = \left(\frac{H_s}{4}\right)^2 \]

For fully developed sea (Pierson-Moskowitz spectrum):

\[ \omega_m = 0.4 \sqrt{\frac{g}{H_s}} \]

JONSWAP Spectrum:

\[ S(\omega) = \frac{\alpha g^2}{\omega^5} \exp\{-1.25 \left(\frac{\omega_p}{\omega}\right)^4\} \cdot \gamma \exp\{-0.5 \left(\frac{\omega-\omega_p}{\sigma \omega_p}\right)^2\} \]

\[ \alpha = 5.061 \left(\frac{\omega_p}{2\pi}\right)^4 H_s^2 \left[1 - 0.287 \log \gamma\right] \]

- \(H_s\): significant wave frequency
- \(\omega_p\): peak frequency
- \(\sigma\): peak enhance coefficient

\[ \sigma = 0.07 \text{ for } \sigma < \omega_p, \text{ and } \sigma = 0.09 \text{ for } \omega \geq \omega_p \]
Combined Sea and Swell:

\[ S(\omega) = S_{swell}(\omega) + S_{sea}(\omega) \]
Response Spectra of a Floating Structure in Irregular Waves

For a linear time-invariant (LTI) system:

\[ H_j(\omega) = \frac{\bar{\zeta}_j(\omega)}{\bar{\eta}(\omega)} \quad j = 1, \ldots, 6 \]

\( H_j(\omega) \): Transfer function (or RAO) of the linear system, determined by the system itself

Spectra of the response:

\[ S_{\zeta_j}(\omega) = |H_j(\omega)|^2 S_\eta(\omega) \], where \( S_\eta(\omega) \) is wave spectrum

Design strategy:

• To avoid large response, make the peak of \( H_j(\omega) \) away from the peak of \( S_\eta(\omega) \)
• For wave energy extraction, make the peak of \( H_j(\omega) \) close to the peak of \( S_\eta(\omega) \)
Short-Term Statistics

- Once the spectrum of a random process is given, the statistics of the random process can be obtained in terms of moments and bandwidth of the spectrum.

Moments:

\[ m_0 = \int_0^\infty S(\omega) d\omega \]
\[ m_2 = \int_0^\infty \omega^2 S(\omega) d\omega \]
\[ m_4 = \int_0^\infty \omega^4 S(\omega) d\omega \]

\[ \ldots \]

Bandwidth coefficient:

\[ \epsilon = \sqrt{1 - \frac{m_2^2}{m_0 m_4}}, \quad 0 \leq \epsilon \leq 1 \]

- If \( \epsilon \geq \sim 0.5 \), called broadband
- If \( \epsilon < \sim 0.5 \), called narrowband

In typical ocean, \( \epsilon = 0.5 \sim 0.6 \)
Issue in Evaluating $m_n$ (with $n \geq 4$)

Typically for ocean waves, we have $S(\omega) \sim \omega^{-5}$ as $\omega \to \infty$

$$m_4 = \int_0^\infty \omega^4 S(\omega) d\omega = \int_0^{\omega_1} \omega^4 S(\omega) d\omega + \int_{\omega_1}^\infty \omega^4 \times \omega^{-5} d\omega$$

\[\rightarrow (\log \omega)|_{\omega_1}^{\infty} \to \infty \text{ Diverges !!}\]

There are two solutions for this issue:

- Truncate the integration up to $3\omega_m$. This has advantage that scaling can be applied between model and full-scale tests.

- Choose a constant upper limit, typically $\omega_{\text{max}} = 2.0 \text{ radian/sec.}$. When scaling, this may be necessary to avoid very large frequency tests.
Zero Up-crossing Period and Peak to Peak Period

Simulation of a random process from a given spectrum $S(\omega)$:

$$\eta(x, t) = \sum_{j}^{N} A_j \cos(\omega_j t - k_j x + \psi_j)$$

$$\omega_j^2 = g k_j \tanh k_j H, \quad A_j^2 = \frac{1}{2} S(\omega_j) \Delta\omega_j$$

$\psi_j$: random phase uniformly distributed in $[0, 2\pi]$.

$T_{zi}$: Zero up-crossing period

$T_{ci}$: Peak to Peak period

Average zero up-crossing period:

$$\bar{T}_z = 2\pi \sqrt{\frac{m_0}{m_2}}$$

Average peak-to-peak period:

$$\bar{T}_c = 2\pi \sqrt{\frac{m_2}{m_4}}$$
How Often is “A” Level Exceeded??

$T(A)$ : Average period for “A” level exceeded

If $A = 0$, then $T_i(A) = T_i(0) = T_{zi} \rightarrow \bar{T}(0) = \bar{T}_z$

$\bar{n}(A) = \frac{1}{\bar{T}(A)}$ : Average frequency of up-crossings if level “A” (i.e. number of up-crossings of level “A” per second)

$\bar{n}(0) = \frac{1}{T_z}$ : Average frequency of zero up-crossings

$\bar{n}(A) = \frac{1}{2\pi} \sqrt{\frac{m_2}{m_0}} e^{-A^2/2m_0} = \frac{1}{T_z} e^{-A^2/2m_0} = \bar{n}(0)e^{-A^2/2m_0}$
Example

A FPSO is exposed to a storm with waves of $m_0 = 4$ meter$^2$ and average period of $T=8$ seconds. Design the free deck height $h$ so that the deck is flooded by green water only once every 10 minutes. (Neglect body motion and diffracted wave effects).

Wave spectrum $S(\omega)$: $m_0 = 4$, $\bar{T}_z = 8$

$$\bar{n}(h) = \frac{1}{\bar{T}_z} e^{-h^2/(2m_0)} = \frac{1}{10 \times 60}$$

$$h = \sqrt{-2m_0 \ln \left( \frac{\bar{T}_z}{600} \right)} = 5.8\text{meters}$$
Maxima

$X_m$: Local maxima of wave elevation

Rayleigh Distribution:

$$p(x_m) = \frac{x_m}{m_0} e^{-x_m^2/(2m_0)}$$

Probability density function

Gaussian distribution $\varepsilon=1$

$$p(x_m) = \frac{1}{\sqrt{2\pi m_0}} e^{-x_m^2/(2m_0)}$$

Rayleigh Distribution $\varepsilon=0$
$a_1, a_2, \ldots, a_n, \ldots$, are the maxima of $\eta(x_0, t)$.

$a^{1/N}$ is the value that is exceeded by $1/N$ th maxima.

Example: $N = 10$, $a^{1/10}$ is the value that is exceeded (on the average) by 10% of the maxima.

$$a^{1/N} = \sqrt{m_0} \sqrt{2 \ln \left( \frac{2\sqrt{1-\epsilon^2}}{1+\sqrt{1-\epsilon^2}} N \right)}$$
1/N th Highest Average Maxima

\( a^{1/N} \): The average value of all the maxima above \( a^{1/N} \).

\[
\overline{a^{1/N}} = E\{a_m | a_m > a^{1/N}\}
\]

This is called the 1/N highest average amplitude.

\[
\overline{a^{1/N}} = 2N \sqrt{m_0} \frac{\sqrt{1-\epsilon^2}}{1+\sqrt{1-\epsilon^2}} \int_{\eta^{1/N}}^\infty \zeta^2 e^{-\zeta^2/2} d\zeta
\]

with \( \eta^{1/N} = a^{1/N} / \sqrt{m_0} \)

\( a^{1/3} \) Significant amplitude: the 1/3 highest average amplitudes

\( H^{1/3} \) Significant height: twice of significant amplitude. \( H^{1/3} = 2a^{1/3} \)

For narrow baned spectrum (\( \epsilon < 0.5 \)), the following holds:

\[
\overline{a^{1/3}} = 2\sqrt{m_0} \quad H_{1/3} = 4\sqrt{m_0}
\]
Example

Given spectrum \( S(\omega) \) with \( m_0=4, \ m_2=1.75, \ m_4=1.0 \), find

Q1: what is the probability that the amplitude is larger than \( A = 10m \)?

\[
\epsilon = \sqrt{1 - \frac{m_2^2}{m_0 m_4}} = \sqrt{1 - \frac{1.75^2}{4 \times 1.0}} = 0.484
\]

\[
P(A > 10m) = \frac{2\sqrt{1-\epsilon^2}}{1+\sqrt{1-\epsilon^2}} e^{-10^2/(2m_0)} = 3.5 \times 10^{-6}
\]

Q2: obtain \( A^{1/10}, \overline{A^{1/10}}, A^{1/3}, H^{1/3} \)

\[
A^{1/10} = \sqrt{2m_0 \ln \left( \frac{2\sqrt{1-\epsilon^2}}{1+\sqrt{1-\epsilon^2}} N \right)} = 4.23\text{meters}
\]

\[
\eta^{1/10} = \frac{A^{1/10}}{\sqrt{m_0}} = 2.12
\]

\[
\overline{A^{1/10}} = \sqrt{m_0} 2N \frac{\sqrt{1-\epsilon^2}}{1+\sqrt{1-\epsilon^2}} \int_{\eta^{1/10}}^{\infty} \zeta^2 e^{-\zeta^2/2} d\zeta = \ldots
\]

\[
A^{1/3} = 2\sqrt[3]{m_0} = 4\text{ meter (assuming narrow band)}
\]

\[
H_{1/3} = 2\overline{A^{1/3}} = 8\text{ meters}
\]