Analytical Mechanics

For holonomic systems
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \]
\[ L = T - V, \quad SW = \sum a_i \dot{q}_i \]

Finding constraint forces using the Lagrangian approach
- Consider \( q_1, \ldots, q_n \) complete but not independent set of coordinates
- They satisfy some holonomic constraint whose constraint forces we seek
- Assume that we have \( m \) holonomic constraints satisfied by these coordinates

\[ \sum_{i=1}^{m} a_{ij} \dot{q}_j + b_i dt = 0 \quad i = 1, \ldots, m \]

Select scalars \( \lambda_i, \quad i = 1, \ldots, m \)
\[ 0 = \sum_{j=1}^{n} a_{ij} \dot{q}_j = 0 \quad \Rightarrow \quad \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{m} a_{ij} \dot{q}_j = 0 \quad (2) \]

By extending Hamilton principle
\[ (3) \quad \int_{t_1}^{t_2} (\dot{q}T + \dot{q}V + \lambda_i \dot{q}_j) dt = 0 \]

Integrate (2) along \((t_1, t_2)\), add to (3):
\[ \int_{t_1}^{t_2} \left( \dot{q}T + \dot{q}V + \sum \lambda_i a_{ij} \dot{q}_j \right) dt = 0 \]

Repeat argument leading to Lagrange's eqs of motion (except for the last step) to obtain:

\[ \sum_{i=1}^{n} \int_{t_1}^{t_2} L - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial q_j} + \sum_{i=1}^{m} \lambda_i a_{ij} \dot{q}_j dt = 0 \]

Idea: \( \lambda_i \) matching \( \lambda_i \) vanish for all \( i \), use \( n+m \) independent \( \dot{q}_j \)'s, AND

Select \( \lambda_1, \ldots, \lambda_m \) in a fashion so that the remaining in brackets vanish

\[ \Rightarrow \left( \int_{t_1}^{t_2} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} \right) = 0, \quad j = 1, \ldots, n \]

Add:
\[ \sum_{j=1}^{n} a_{ij} \dot{q}_j + b_i = 0 \quad i = 1, \ldots, m \]

\[ \frac{d}{dt} a_{ij} \dot{q}_j + b_i = 0 \quad i = 1, \ldots, m \]

\( \lambda_i \): Lagrangean multipliers
\[ n+m \text{ eqs, } n+m \text{ unknowns} \]
NOTE:
\[ K_j = \sum_j \alpha_{ij} \]

is the \( j \) coordinate force result

Also, the above formulation covers non-holonomic systems as well because constraints can also be written in the form (11)

Example
Reconsider "Glider sliding on a pendulum under the effect of follower force"

\[ \theta \]

\[ \text{Question: Constraint force } N? \]

\[ \text{Select: } q_1 = r \]

\[ q_2 = \theta \]

\[ q_3 = \phi \]

angle of beam with horizontal

\[ n = 3 \]

\[ \text{Constraint: } q_2 - q_3 = 0 \quad (m = 1) \]

\[ \alpha_{nl} = 0; \quad \alpha_{l2} = 1; \quad \alpha_{l3} = -1 \]

Active generalized forces unrelated to constraints (non-potential)

in \( q \), direction:

\[ q_1 = S \]

\[ q_2 = 0 \]

\[ q_3 = F \theta \sin \theta \]

\[ \alpha_{nl} \]

\[ n = 3 \]

only one lagrangian multiplier \( \alpha \):

\[ L = T - V \]

\[ T = T_{beam} + T_{Glor} = \frac{1}{2} M \dot{\theta}^2 + \frac{1}{2} m (\dot{r}^2 + \dot{\theta}^2) \]

\[ V = V_{beam} + V_{Glor} = -M g \frac{1}{2} \cos \theta - m g r \cos \theta \phi \]

\[ L = \frac{1}{2} M \dot{\theta}^2 + \frac{1}{2} m (\dot{r}^2 + \dot{\theta}^2) + M g \frac{1}{2} \cos \theta - m g r \cos \theta \phi \]

Equation of motion:

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \lambda_1 \alpha_{11} \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = \lambda_1 \alpha_{12} = \lambda_1 \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \phi} \right) - \frac{\partial L}{\partial \phi} = \Phi \theta \sin \theta + \lambda_1 \alpha_{13} \]

\[ \lambda_1 = \frac{d}{dt} \left( \frac{\partial L}{\partial \theta} \right) + m g r \sin \theta \]

\[ \lambda_1 = \frac{d}{dt} \left( m \dot{r} \dot{\theta} \right) + m g r \sin \theta \]

To obtain \( N \): non-potential force

\[ \delta W = \left( \frac{\partial}{\partial \theta} \right) \delta q + \left( \frac{\partial}{\partial \phi} \right) \delta \phi \]

\[ N = \frac{\partial V}{\partial \theta} \]

\[ \text{friction follower force} \]

\[ K_j \]
\[ N = \frac{1}{2} K s = 2 m r \dot{\theta} + m r \ddot{\theta} + m g x \sin \theta \]