II. Dynamics of systems of particles

\[ F_i = F_i^{\text{ext}} + F_i^{\text{int}} \]

\[ F_i^{\text{int}} = \sum_{j \neq i}^n k_{ij} \]

Newton III: \[ k_{ij} + k_{ji} = 0 \]

Also: \[ \sum_{i=1}^n E_i \times F_i^{\text{int}} = 0 \]

Example: For two masses

\[ E_i \times k_{ij} + E_j \times k_{ji} = \]

\[ = (E_i - E_j) \times k_{ji} = 0 \]

Constraints: Geometric limitation on the absolute or relative motion of particles

\( \mathbf{r}_i = (x_i, y_i, z_i) \to \mathbf{r}_i + \mathbf{u}_i = 0 \)

\( \mathbf{u}_i \) is prescribed

Case: Can eliminate \( \mathbf{u}_i \) and get 2 constraints

3. Smooth plane \( \mathbf{z}_i = 0 \to 1 \) constraint

4. Spherical non-rolling \( \mathbf{z}_i = \text{const} \)

Degrees of Freedom
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\[ \# \text{DOF} = 3n - (\# \text{of independent scalar restrictions on position}) \]

or

\[ \# \text{of constraints} \]

Total mass

\[ M = \sum_{i=1}^{n} m_i \]

Center of mass: geometric point w.r.t. which the total mass moment is zero

\[ \sum_{i=1}^{n} m_i (x_i - x_c) = 0 \]

(a) Linear momentum principle

\[ \dot{\mathbf{P}} = \mathbf{F}_{\text{ext}} \quad \mathbf{F}_{\text{ext}} = \sum_{i=1}^{n} \mathbf{F}_i \]

Define:

\[ \mathbf{P} = \sum_{i=1}^{n} \mathbf{p}_i = M \frac{d}{dt} \sum_{i} m_i \mathbf{v}_i = M \frac{d}{dt} \left( \sum_{i} m_i \mathbf{v}_i \right) = M \dot{\mathbf{v}}_c = MV_c \]

Do \[ \sum \text{on (r)} \]

\[ \dot{\mathbf{P}} = \mathbf{F}_{\text{ext}} = \sum_{i=1}^{n} \mathbf{F}_i = \mathbf{F}_{\text{ext}} \]

Linear momentum principle

\[ \text{if } \mathbf{F}_{\text{ext}} = \sum_{i} \mathbf{F}_i = \mathbf{0} \implies \mathbf{P} = \text{constant} \quad \text{(Conservation of linear momentum)} \]

(b) Angular momentum principle

\[ \dot{\mathbf{P}} = \mathbf{F}_{\text{int}} = \mathbf{F}_{\text{int}} \]

\[ \quad \quad \sum_{i} \mathbf{p}_i \times \dot{\mathbf{p}}_i = \sum_{i} \mathbf{p}_i \times \frac{d}{dt} \mathbf{p}_i - \sum_{i} \mathbf{p}_i \times \dot{\mathbf{p}}_i = \mathbf{0} \]

\[ = \mathbf{0} \]

\[ = \mathbf{H}_\theta + \mathbf{v}_e \times \mathbf{P} \]

\[ \implies \mathbf{H}_\theta + \mathbf{v}_e \times \mathbf{P} = \mathbf{M}_\theta \quad \mathbf{M}_\theta = \sum_{i=1}^{n} \mathbf{p}_i \times \mathbf{E}_i \]
c) Work-Energy Principle

Define

\[ W_{12} = \sum_i w_i \]
\[ T = \sum_i T_i \]

Therefore, by the Work-Energy principle, includes \( W_{12}^{\text{int}} \)

\[ W_{12} = \sum_i F_i \cdot \dot{r}_i + \sum_i F_i^{\text{int}} \cdot \dot{r}_i \]
\[ = \sum_i \int_{t_1}^{t_2} F_i^{\text{int}} \dot{r}_i \ dt + \int_{t_1}^{t_2} \sum_i (K_{ij} \dot{V}_i + K_{ji} \dot{V}_j) \ dt \]
\[ = \int_{t_1}^{t_2} \sum_{i,j} K_{ij} (\dot{V}_i - \dot{V}_j) \ dt \]

Important special case:

\[ K_{ij} (\dot{V}_i - \dot{V}_j) = 0 \quad \text{for all } i,j \]

\[ \Rightarrow (\dot{X}_i - \dot{X}_j) = (\dot{V}_i - \dot{V}_j) \]

\[ \frac{d}{dt} (r - r_j) = 0 \quad \Rightarrow \quad \frac{d}{dt} (r - r_j)^2 = 0 \quad \text{for all } i,j \]

Definition: Systems of particles with \( r_i - r_j \) = \text{const} are called rigid body.

For such systems:

\[ W_{12}^{\text{int}} = T_2 - T_1 \]
If furthermore all external forces are potential, i.e.,

\[ F^\text{ext}_i = -\nabla V_i(x_j, t) \]

Then

\[ T_2 - T_1 = W_1 + \int \sum_i F^\text{ext}_i \, ds = \sum V_i^1 - V_i^2 = \bar{V} - \bar{V} \]

where

\[ \bar{V} = \sum V_i^1 \]

\[ V_2 = \sum V_i^2 \]

\[ \bar{T} + \bar{V} = \text{Const} \]

Conservation of Energy. (rigid body) external forces are potential.

Example

Assume \( \dot{r}(0) = 0 \)

\( r(0) = r_0 \)

Question: minimum value of \( r \)

maximum value string force