Recall:
\[ E = T + V = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} mr^2 \dot{\phi}^2 + mg r = E_0 = \text{Const} \]

\[ H_0 = mr^2 \dot{\phi} = \text{Const} \quad \Rightarrow \quad \dot{\phi} = \frac{H_0}{mr^2} \]  \( \quad (\star) \)

Note: \( \phi \) is a cyclic coordinate (ignorable)
\[ \frac{\partial E}{\partial \phi} = 0 \]

When such a coordinate present,
\[ \# \text{DOF can be reduced by one} \quad \Rightarrow \quad \text{reduced mechanical system} \]

In present case, use \((\star)\) to obtain reduced energy
\[ E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} mr^2 \dot{\phi}^2 + mg r = E_0 \]
\[ T(r) \quad V(r) \]
\[ \dot{r} = \sqrt{\frac{I}{m} (E_0 - V(r))} \]

**Rigid Body Dynamics**

1. \[ \Delta \text{AB} = \text{Const} \]

2. \[ \# \text{DOF} = 6 \]

3. Velocities at different points of a rigid body
\[ \vec{V}_B = \vec{V}_A + \vec{\omega} \times \vec{r}_{AB} \]
\[ \vec{V}_B = \vec{V}_A + \vec{\omega} \times \vec{r}_{AB} \]

It turns out that there exist a unique vector \( \vec{\omega} \) (angular velocity of the rigid body) such that
\[ \vec{V}_B = \vec{V}_A + \vec{\omega} \times \vec{r}_{AB} \]  \[ \text{for all } A \in \mathbb{B} \]
a) If the rotation of the rigid body can be instantaneously decomposed to a finite # of rotations about "well-understood" fixed axes, then the angular velocities defined for those rotations, then $\omega$ is just the sum of those angular velocities.

\[ \sum_{i=1}^{n} \omega_i \rightarrow \omega = \sum_{i=1}^{n} \omega_i \]

Surprising, because finite rotation in 3D doesn't commute to prove (i), note that instantaneously, $\Omega$ performs an instantaneous rotation about $\hat{A}$

In general, rotation in 3D about a fixed point can be described through matrix multiplication.

\[ e(\tau) = R(\tau) e_0 \]

where $R(\tau)$ is a proper orthogonal matrix

Main properties of such matrices:

(i) Preserve length $\| e(\tau) \| = \| e_0 \|$, or $\langle R(\tau) e_0, R(\tau) e_0 \rangle = \langle e_0, e_0 \rangle$

In general $\langle I a, b \rangle = \langle a, I^T b \rangle$ (transpose of $I$)

\[ \langle e_0, R(\tau) e_0 \rangle = \langle e_0, e_0 \rangle \]

$I$ (because $e_0$ is an identity)

here $I = (1 \ 0 \ 0)
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow R^{-1} = R^T$

\[ \text{det}(R^T) \cdot \text{det}(R) = 1 \Rightarrow |\text{det}(R)| = 1 \]

b) Preserve orientation of vectors

$\Rightarrow \text{det} R > 0 \Rightarrow |\text{det}(R(\tau))| = 1$
Example

\[ R(t) = \begin{pmatrix} \cos \theta(t) & \sin \theta(t) & 0 \\ -\sin \theta(t) & \cos \theta(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{pmatrix} = \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{pmatrix} \]

Using the above, fixing \( A \) we obtain

\[ \frac{d}{dt} \{ \dot{x}_{AB}(t) = \dot{R}(t) x_{AB}(t) \} \]

\[ \dot{x}_{AB} = \dot{R} x_{AB}(t) \]

Note: \( \dot{R} R^T = I \)  / \( \frac{d}{dt} \)

\[ \dot{R} R^T + R \dot{R}^T = 0 \]

\[ \implies \dot{R} R^T = -R \dot{R}^T R \]

\[ \implies \dot{x}_{AB} = \begin{pmatrix} \xi_{AB}^A(t) \\ \xi_{AB}^B(t) \\ \xi_{AB}^C(t) \end{pmatrix} \]

\[ = \begin{pmatrix} \xi_{AB}^A(t) \\ \xi_{AB}^B(t) \\ \xi_{AB}^C(t) \end{pmatrix} = \begin{pmatrix} \xi_{AB}^A(t) \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \]

## Skew Symmetric

By Assignment #2, for any 3D skew symmetric matrix \( \omega^3 \), there exists a 3D vector \( \omega^A \) such that

\[ \omega^A \times \omega^B = \omega^A \times \omega^B \]

\[ \dot{x}_{AB} = \dot{R}(t) x_{AB}(t) = \omega(t) \times x_{AB}(t) \]

\[ \implies \dot{y}_0 = y_0 + \omega_A \times y_{AB} \]

\[ x_0 \]

\[ \begin{pmatrix} A \end{pmatrix} \]

\[ \begin{pmatrix} y_0 \end{pmatrix} \]