To complete the argument started last time we need to show that \( \omega^A \) is in fact independent of \( A \).

\[
\begin{align*}
\omega^A \times (I_{AB} + I_{AC}) & = \omega^B \times I_{BC} \\
(\omega^A - \omega^B) \times I_{BC} & = 0 \quad \text{because } I_{BC} \text{ is arbitrary} \quad \text{(i)} \\
\Rightarrow \quad \omega^A & = \omega^B \\
\Rightarrow \quad \omega^B & = I_{AB} + \omega \times I_{AB}
\end{align*}
\]

(2) Show: \( \omega \) can be obtained by adding angular velocities about different axes.

To see this, fix \( A \) instantaneously and consider composition of \( k \) rigid body rotations about \( A \):

\[
\begin{align*}
\mathbf{R}_{AB}(t) & = R_z(t) \ R_{AB}(0) \\
& = \begin{pmatrix} B_k & \ldots & B_2 & B_1 \end{pmatrix} \ \mathbf{R}_{AB}(0) \\
& \quad (k \ \text{rotation})
\end{align*}
\]

\[
\dot{\mathbf{R}}_{AB} = \begin{pmatrix} B_k & \ldots & B_2 & B_1 \end{pmatrix} \ \dot{\mathbf{R}}_{AB}(0) \\
\begin{cases}
+ B_k \ B_{k+1} \ \dddot{B}_1 \\
+ \dddot{B}_k \ B_{k-1} \ \dddot{B}_1 \\
\end{cases}
\]

Recall:

\[
\dot{B}_i = -B_2 \ \dddot{B}_2 \\
\mathbf{I}_{AB} = \begin{pmatrix} -B_k \ B_k \ B_k \ \ldots \ B_1 + \ldots \\
+ B_k \ B_k \ B_k \ \ldots \ B_1 \ B_1 \end{pmatrix}
\]

Note:

\[
-\mathbf{B}_B \ B_B^T \mathbf{I}_{AB} = \mathbf{I}, \quad \mathbf{B}_B(t) = \mathbf{I}
\]
\[
\mathbf{\omega} = \mathbf{\omega}_k \times (\mathbf{e}_1 \times \mathbf{e}_2)
\]

Conclusion:

\[
\mathbf{\omega} = \sum_{i=1}^{K} \mathbf{\omega}_i
\]

In application, how do we find \(\mathbf{\omega}\)?

a) Identify all axes about which body rotates then add component angular velocities.

b) If we know \(\mathbf{u}\) and \(\mathbf{v}\) we obtain \(\mathbf{\omega}\) by solving solution at first.

\[
\mathbf{v} - \mathbf{u} = \mathbf{\omega} \times \mathbf{r}_{uv}
\]

c) In two-D motion \((x-y)\) plane \(\mathbf{\omega} = \mathbf{\omega}_x\)

Then \(\mathbf{\omega}\) can be identified as follows.

E.g.

\[
\mathbf{\omega} = (\alpha + \omega) \mathbf{k}
\]

One last word about rigid body rotation.

Often some vectors are more convenient in frame that rotating with rigid body. E.g.

In this example, as soon later, the angular momentum vector is easier to compute in a rotating frame \((\xi, \eta, \zeta)\).

For cases like this we need to know how to evaluate the "real" relative to inertial frame time derivative of vectors in question.
\[ \dot{\mathbf{w}} = \dot{u} + \omega \times u \]

also \[ \dot{w} = \dot{v} + \dot{u} + \omega \times u \]

**Example (2D)**

\[ V_B = ? \]

Its direction is known to be parallel to the line AB

\[ \# \text{ DOF} = 3 \times 3 - 2 - 2 - 2 \]

\[ \text{Joint A} \quad \text{Joint B} \quad \text{Joint C} \quad \text{Constraint} \]

\[ \text{Joint at } 2 \text{ rad} \]

\[ \text{is Constrained to point} \]

\[ \text{B of rail } l \]

\[ \dot{v}_B \text{ Can be expanded in terms of} \]

Say \( \alpha \) and \( \dot{\alpha} (= \ddot{\alpha}) \) and \( V_0 \)

\[ V_0 = \dot{V}_A + \omega_1 \times \dot{X}_{AB} \]

\[ V_B = \dot{V}_C + \omega_2 \times \dot{X}_{CD} \]

\[ \omega_1 \]

\[ \omega_1 = -\dot{\alpha} \hat{y} \]

\[ \omega_2 = \dot{\beta} \hat{y} \]

From geometry \[ 2ab \sin \beta - a = b \sin \theta \]

\[ \dot{V}_B = V_0 \dot{a} \frac{\sqrt{a^2 + b^2 \sin^2 \theta}}{a \sqrt{a^2 + b^2 \sin^2 \theta} + \alpha (\dot{\alpha}^2 \sin \theta + \ddot{\alpha}^2 \cos \theta)} \]