“Examinations are formidable even to the best prepared, for
the greatest fool may ask more than the wisest person may
answer.”

Charles Caleb Colton (1780-1832)
Problem 1: (Knolwes 3.24) A tensor $T$ is symmetric, orthogonal and positive definite. Determine $T$.

Problem 2: (Essentially Knowles 1.18) Let $R$ be the 3-dimensional Euclidean vector space of polynomials of degree not exceeding two, where the scalar product between two vectors $f = f(t)$ and $g = g(t)$ is defined by

$$f \cdot g = \int_{-1}^{1} f(t)g(t) \, dt.$$ 

i) Show that $f_1 = 1$, $f_2 = t$, $f_3 = t^2$ is a basis for $R$.

ii) Find an orthonormal basis for $R$.

Problem 3: (Based on Knowles 3.17) Let $A$ and $B$ be two symmetric tensors whose matrices of components in an orthonormal basis $\{e_1, e_2\}$ are

$$[A] = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad [B] = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$$

respectively.

i) Do $A$ and $B$ have a common principal basis?

ii) Determine a principal basis for $A$.

Problem 4: (Essentially Knowles 3.18) If a tensor $P$ has the property $PP^T = I$ show that

i) $P$ is nonsingular,

ii) $P^T P = I$, and

iii) $P$ is orthogonal, i.e. that $P$ preserves length ($\iff |Px| = |x|$ for all vectors $x$).
Problem 5: (Essentially Knowles 3.10, 3.26). Let $A$ be a skew-symmetric tensor on a finite dimensional Euclidean vector space.

i) If $A$ has a real eigenvalue $\alpha$, show that $\alpha = 0$.

ii) Show that $I + A$ and $I - A$ are both nonsingular tensors.

iii) Show that $I + A$ and $I - A$ commute, i.e. that $(I + A)(I - A) = (I + A)(I - A)$.

iv) Show that $(I - A)(I + A)^{-1}$ is an orthogonal tensor.

Problem 6: (Essentially Knowles 2.18) Let $A$ be a symmetric tensor on an $n$-dimensional Euclidean vector space. Suppose that $A$ has distinct eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$ and a corresponding set of orthonormal eigenvectors $a_1, a_2, \ldots, a_n$.

i) For any positive integer $m$, show that $A^m$ (defined as $\underbrace{AA\ldots A}_{m\text{ times}}$) has eigenvalues $\alpha_1^m, \alpha_2^m, \ldots, \alpha_n^m$ and corresponding eigenvectors $a_1, a_2, \ldots, a_n$.

ii) Let $p(x) = \sum_{k=0}^{n} c_k x^k$ be an arbitrary polynomial of degree $n$ where the $c$’s are real numbers. Let $P$ be the tensor defined by $P = P(A) = \sum_{k=0}^{n} c_k A^k$ where $A^0 = I$. Show that the eigenvalues of $P$ are $p(\alpha_1), p(\alpha_2), \ldots, p(\alpha_n)$ and that the corresponding eigenvectors are $a_1, a_2, \ldots, a_n$.

iii) Consider the special case where $p(x)$ is the characteristic polynomial of $A$, i.e. $p(x) = \det[A - xI]$. Show that that the corresponding tensor $P(A)$ is the null tensor $O$. 