2.035: Midterm Exam - Part 1
Spring 2007
SOLUTION

PROBLEM 1:

a) A vector space is a set \( V \) of elements called vectors together with operations of addition and multiplication by a scalar, where these operations must have the following properties:

(A) Corresponding to every pair of vectors \( \mathbf{x}, \mathbf{y} \in V \) there is a vector in \( V \), denoted by \( \mathbf{x} + \mathbf{y} \), and called the sum of \( \mathbf{x} \) and \( \mathbf{y} \), with the following properties:

\[
\begin{align*}
(1) \quad & \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x} \text{ for all } \mathbf{x}, \mathbf{y} \in V; \\
(2) \quad & \mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z} \text{ for all } \mathbf{x}, \mathbf{y}, \mathbf{z} \in V; \\
(3) \quad & \text{there is a unique vector in } V, \text{ denoted by } \mathbf{o} \text{ and called the null vector, with the property that } \mathbf{x} + \mathbf{o} = \mathbf{x} \text{ for all } \mathbf{x} \in V; \text{ and}
\end{align*}
\]

(B) Corresponding to every real number \( \alpha \in \mathbb{R} \) and every vector \( \mathbf{x} \in V \) there is a vector in \( V \), denoted by \( \alpha \mathbf{x} \), and called the product of \( \alpha \) and \( \mathbf{x} \), with the following properties:

\[
\begin{align*}
(5) \quad & \alpha(\beta \mathbf{x}) = (\alpha\beta)\mathbf{x} \text{ for all } \alpha, \beta \in \mathbb{R} \text{ and all } \mathbf{x} \in V; \\
(6) \quad & \alpha(\mathbf{x} + \mathbf{y}) = \alpha \mathbf{x} + \alpha \mathbf{y} \text{ for all } \alpha \in \mathbb{R} \text{ and all } \mathbf{x}, \mathbf{y} \in V; \\
(7) \quad & (\alpha + \beta)\mathbf{x} = \alpha \mathbf{x} + \beta \mathbf{x} \text{ for all } \alpha, \beta \in \mathbb{R} \text{ and all } \mathbf{x} \in V; \text{ and}
\end{align*}
\]

(8) \( 1\mathbf{x} = \mathbf{x} \) for all \( \mathbf{x} \in V \).

b) A set of vectors \( \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\} \) is said to be linearly independent if the only scalars \( \alpha_1, \alpha_2, \ldots, \alpha_n \) for which

\[
\alpha_1\mathbf{f}_1 + \alpha_2\mathbf{f}_2 + \cdots + \alpha_n\mathbf{f}_n = \mathbf{o}
\]

are \( \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0 \).

c) If a vector space \( V \) contains a linearly independent set of \( n \) (\( n > 0 \)) vectors but contains no linearly independent set of \( n + 1 \) vectors we say that the dimension of \( V \) is \( n \).

d) If \( V \) is a \( n \)-dimensional vector space then any set of \( n \) linearly independent vectors is called a basis for \( V \).

e) If \( \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\} \) is a basis for an \( n \)-dimensional vector space \( V \), then any vector \( \mathbf{x} \in V \) can be expressed in the form

\[
\mathbf{x} = \xi_1\mathbf{f}_1 + \xi_2\mathbf{f}_2 + \cdots + \xi_n\mathbf{f}_n
\]

where the set of scalars \( \xi_1, \xi_2, \ldots, \xi_n \) is unique and are called the components of \( \mathbf{x} \) in the basis \( \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\} \).
f) To every pair of vectors \( \mathbf{x}, \mathbf{y} \in \mathbb{V} \) we associate a real number denoted by \( \mathbf{x} \cdot \mathbf{y} \) and called the scalar product of \( \mathbf{x} \) and \( \mathbf{y} \) provided that this product has the following properties:

(9) \( \mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x} \) for all \( \mathbf{x}, \mathbf{y} \in \mathbb{V} \);

(10) \( (\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z} \) for all \( \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{V} \);

(11) \( (\alpha \mathbf{x}) \cdot \mathbf{y} = \alpha (\mathbf{x} \cdot \mathbf{y}) \) for all \( \alpha \in \mathbb{R} \) and all vectors \( \mathbf{x}, \mathbf{y} \in \mathbb{V} \); and

(12) \( \mathbf{x} \cdot \mathbf{x} > 0 \) for all vectors \( \mathbf{x} \neq \mathbf{o} \) in \( \mathbb{V} \).

g) The real number denoted by \( |\mathbf{x} - \mathbf{y}| \) and defined as \( |\mathbf{x} - \mathbf{y}| = \left( (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \right)^{1/2} \) is called the distance between the vectors \( \mathbf{x} \) and \( \mathbf{y} \).

h) If \( \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} \) is a basis for an \( n \)-dimensional vector space and if

\[
\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 
0 & \text{if } i \neq j, \\
1 & \text{if } i = j 
\end{cases} \quad i, j = 1, 2, \ldots, n,
\]

we say that \( \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} \) is an orthonormal basis.

i) A linear transformation \( \mathbf{A} \) on a vector space \( \mathbb{V} \) is a transformation that assigns to each vector \( \mathbf{x} \in \mathbb{V} \) a unique vector in \( \mathbb{V} \) which we denote by \( \mathbf{A}\mathbf{x} \) with the properties:

(13) \( \mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y} \) for all vectors \( \mathbf{x}, \mathbf{y} \in \mathbb{V} \); and

(14) \( \mathbf{A}(\alpha \mathbf{x}) = \alpha \mathbf{A}\mathbf{x} \) for every \( \alpha \in \mathbb{R} \) and every vector \( \mathbf{x} \in \mathbb{V} \).

j) Let \( \mathcal{S} \) be a subset of a vector space \( \mathbb{V} \). Suppose further that \( \mathcal{S} \) itself is in fact a vector space on its own right under the same operations of addition and scalar multiplication as in \( \mathbb{V} \). Then \( \mathcal{S} \) is said to be a subspace of \( \mathbb{V} \). Finally, suppose in addition that \( \mathbf{A}\mathbf{x} \in \mathcal{S} \) for all \( \mathbf{x} \in \mathcal{S} \). Then we say that \( \mathcal{S} \) is an invariant subspace of \( \mathbf{A} \).

k) The set \( \mathcal{N} \) of all vectors \( \mathbf{x} \) for which \( \mathbf{A}\mathbf{x} = \mathbf{o} \) is called the null space of \( \mathbf{A} \).

l) A linear transformation \( \mathbf{A} \) is said to be singular if there is a vector \( \mathbf{x} \neq \mathbf{o} \) for which \( \mathbf{A}\mathbf{x} = \mathbf{o} \).

m) The \( n^2 \) real numbers \( A_{ij} \) defined by

\[
A_{ij} = \mathbf{e}_j \cdot \mathbf{A}\mathbf{e}_i
\]

are called the components of the linear transformation \( \mathbf{A} \) in the basis \( \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} \).

n) A scalar valued function \( \phi \) defined on the set of all linear transformations is said to be a scalar invariant if \( \phi(Q\mathbf{A}Q^T) = \phi(\mathbf{A}) \) for every linear transformation \( \mathbf{A} \) and all orthogonal linear transformations \( Q \).
PROBLEM 2:

a) Consider the set \( V \) of all \( 2 \times 2 \) matrices \( \mathbf{x} \) of the form

\[
\mathbf{x} = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}
\]

where \( x_1 \) and \( x_2 \) range over all real numbers; let

\[
\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

be the null vector; and define addition, \( \mathbf{x} + \mathbf{y} \), and scalar multiplication, \( \alpha \mathbf{x} \), in the natural way by

\[
\begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \\ y_2 & y_1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 & x_2 + y_2 \\ x_2 + y_2 & x_1 + y_1 \end{pmatrix}, \quad \alpha \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1 & \alpha x_2 \\ \alpha x_2 & \alpha x_1 \end{pmatrix}.
\]

One can verify that all of the requirements (1)–(8) of Problem 1 are satisfied by these operations, and moreover, that \( \mathbf{x} + \mathbf{y} \) and \( \alpha \mathbf{x} \) are both in \( V \) when \( \mathbf{x}, \mathbf{y} \in V \) and \( \alpha \in \mathbb{R} \). Thus \( V \) is a vector space.

b) Consider the following two vectors \( \mathbf{f}_1 \) and \( \mathbf{f}_2 \):

\[
\mathbf{f}_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.
\]

One can readily verify that if \( \alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2 = \mathbf{0} \), then necessarily \( \alpha_1 + 2\alpha_2 = 0 \) and \( 2\alpha_1 + \alpha_2 = 0 \) which in turn implies that \( \alpha_1 = \alpha_2 = 0 \). Thus \( \{ \mathbf{f}_1, \mathbf{f}_2 \} \) is a linearly independent set of vectors.

c) Consider the following three vectors \( \mathbf{f}_1, \mathbf{f}_2, \mathbf{x} \),

\[
\mathbf{f}_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}.
\]

where \( \mathbf{x} \) is an arbitrary vector in \( V \). One can readily verify that if \( \alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2 + \alpha_3 \mathbf{x} = \mathbf{0} \) then necessarily \( \alpha_1 + 2\alpha_2 + x_1\alpha_3 = 0 \) and \( 2\alpha_1 + \alpha_2 + x_2\alpha_3 = 0 \). Observe that the choice

\[
\alpha_1 = \frac{1}{3}(2x_2 - x_1), \quad \alpha_2 = \frac{1}{3}(2x_1 - x_2), \quad \alpha_3 = -1
\]

satisfies these two scalar equations. Thus if \( \alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2 + \alpha_3 \mathbf{x} = \mathbf{0} \) this does not require that all the \( \alpha \)'s vanish and so \( \{ \mathbf{f}_1, \mathbf{f}_2, \mathbf{x} \} \) is a linearly dependent set of vectors. Recall that \( \{ \mathbf{f}_1, \mathbf{f}_2 \} \) is a linearly independent set of vectors. Thus the dimension of \( V \) is 2.

d) Since \( V \) is a 2-dimensional vector space and since the set of vectors \( \{ \mathbf{f}_1, \mathbf{f}_2 \} \) is linearly independent, it follows that \( \{ \mathbf{f}_1, \mathbf{f}_2 \} \) is a basis for \( V \).

e) Consider the basis \( \{ \mathbf{f}_1, \mathbf{f}_2 \} \) and let \( \mathbf{x} \) be an arbitrary vector in \( V \). Then one can readily verify that

\[
\mathbf{x} = \xi_1 \mathbf{f}_1 + \xi_2 \mathbf{f}_2 \quad \text{where} \quad \xi_1 = \frac{1}{3}(2x_2 - x_1) \quad \text{and} \quad \xi_2 = \frac{1}{3}(2x_1 - x_2)
\]

are the components of \( \mathbf{x} \) in the basis \( \{ \mathbf{f}_1, \mathbf{f}_2 \} \).
f) Corresponding to any two vectors $x, y \in V$, where
\[
x = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_1 \end{pmatrix},
\]
tentatively define their scalar product as
\[
x \cdot y = x_1y_1 + x_2y_2.
\]
One can verify that this definition satisfies all of the requirement (9)–(12) of Problem 1 and therefore is in fact a legitimate definition of a scalar product.

g) The distance between the two vectors
\[
x = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_1 \end{pmatrix}
\]
is
\[
|x - y| = \left( (x - y) \cdot (x - y) \right)^{1/2} = \left( (x_1 - y_1)^2 + (x_2 - y_2)^2 \right)^{1/2}.
\]
h) Consider the two vectors
\[
e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Observe that $e_1 \cdot e_2 = 0, |e_1| = |e_2| = 1$ and so $\{e_1, e_2\}$ forms an orthonormal basis for $V$.

i) Consider a transformation $A$ that takes the vector
\[
x = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}
\]
into the vector $Ax = \begin{pmatrix} x_2 & x_1 \\ x_1 & x_2 \end{pmatrix}$

One can verify that the requirements (13), (14) of Problem 1 are satisfied, and moreover that $Ax \in V$ for all $x \in V$. Therefore $A$ is a linear transformation.

j) Consider the set $S$ of all vectors $x$ of the form
\[
x = \begin{pmatrix} x & x \\ x & x \end{pmatrix}
\]
where $x$ ranges over all real numbers. Clearly $S$ is a subset of $V$. Moreover, one can verify that $S$ itself is a vector space on its own right under the same operations of addition and scalar multiplication as in $V$. Thus $S$ is a subspace of $V$. Furthermore, observe that $Ax = x$ for all vectors $x \in S$, so that in particular $Ax \in S$ for all $x \in S$. Thus $S$ is an invariant subspace of $A$. (In fact it is a one-dimensional invariant subspace associated with the eigenvalue $+1$).

k) From item (i) we see that if $Ax = o$ then necessarily $x = o$. Thus the null space of $A$ is comprised of a single vector, the null vector: $N = \{o\}$.

l) As noted in the preceding item, $Ax = o$ implies that necessarily $x = o$. Therefore $A$ is nonsingular.
m) Observe from the definitions of \( A, e_1 \) and \( e_2 \) that \( A e_1 = e_2 \) and \( A e_2 = e_1 \). Thus the components of \( A \) in the basis \( \{ e_1, e_2 \} \) are

\[
A_{11} = e_1 \cdot A e_1 = e_1 \cdot e_2 = 0, \quad A_{12} = e_1 \cdot A e_2 = e_1 \cdot e_1 = 1,
\]

\[
A_{22} = e_2 \cdot A e_2 = e_2 \cdot e_1 = 0, \quad A_{21} = e_2 \cdot A e_1 = e_2 \cdot e_2 = 1.
\]

n) Consider the scalar-valued function \( \phi(A) = \det A \) defined for all linear transformations \( A \). Then for any linear transformation \( A \) and any orthogonal transformation \( Q \) we have

\[
\phi(QA Q^T) = \det(QA Q^T) = \det(Q) \det(A) \det(Q^T) = \det(Q) \det(A) \det(Q) = (\pm 1)^2 \det A = \det A.
\]

Thus the function \( \phi(A) = \det A \) has the property that \( \phi(QA Q^T) = \phi(A) \) for every linear transformation \( A \) and all orthogonal linear transformations \( Q \). Thus \( \det A \) is a scalar invariant of \( A \).