Supplementary note on surface tension

The following figure is from White (e7), Fig. 11 in page 33.

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In this figure, the surface tension is notated by $\Upsilon$ which is $\sigma$ in the lecture.

(b) is the same figure for a drop discussed in the lecture.

The net force balance is

\[ P_i A - P_a A = \sigma 2\pi R \]
\[ \Delta P = P_i - P_a = \frac{2\sigma}{R} \]

(c) is a more a general case.

The force by pressure differences $\Delta P$ ($= P_i - P_a$) is

\[ F_{\Delta P} = (P_i - P_a) dL_1 dL_2 \]

Note, here the area, $dL_1 dL_2$ is not exactly the projected area, perpendicular to the direction in which we consider the force balance. But, as we consider a very small element, the area would be very close to $dL_1 dL_2$.

And, $dL_1 = R_1 2 d\phi_1$ and $dL_2 = R_2 2 d\phi_2$

Now,

\[ F_{\Delta P} = (P_i - P_a) dL_1 dL_2 = (P_i - P_a)(R_1 2 d\phi_1)(R_2 2 d\phi_2) \]

The force by the surface tension is

\[ F_\sigma = 2(\sigma dL_1 \sin d\phi_2) + 2(\sigma dL_2 \sin d\phi_1) \]

Since $d\phi_1$ is very small ($d\phi_1 \ll 1$), $\sin d\phi_1 = d\phi_1$

Similarly, $\sin d\phi_2 = d\phi_2$

Then, the force by surface tension becomes

\[ F_\sigma = 2(\sigma dL_1 \sin d\phi_2) + 2(\sigma dL_2 \sin d\phi_1) = 4\sigma R_1 d\phi_1 d\phi_2 + 4\sigma R_2 d\phi_2 d\phi_1 \]

Net force balance gives

\[ F_{\Delta P} = F_\sigma \]
\[ (P_i - P_a) 4R_1 R_2 d\phi_1 d\phi_2 = 4\sigma d\phi_1 d\phi_2 (R_1 + R_2) \]
\[ P_i - P_a = \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]